Abstract

We discuss a class of graphs called "perfect graphs." After defining them and getting intuition with a few simple examples (and one less simple example), we present a proof of the Weak Perfect Graph Theorem.

The presentation of the Weak Perfect Graph Theorem, especially the intuition for duplication, is based on Diestel's presentation [1]. Historical notes and basic graph terminology come from West's textbook [5].

1 Perfect Graphs

A graph G is **perfect** if for every induced subgraph H of G, the chromatic number of H is equal to the size of the largest clique of H.

Let's fix some notation: $\chi(G)$ is the chromatic number of G, $\omega(G)$ is the size of the largest clique of G, and $\alpha(G)$ is the size of the largest independent set of G. In our new notation, a graph is perfect if for every induced subgraph H: $\chi(H) = \omega(H)$. We use the notation $N_G(S)$ to mean the neighbors of a set of vertices S in the graph G. When G is clear from context we omit the subscript. The complement of G is another graph $\overline{G}$ on the same vertex set; for every pair of vertices $\{u, v\}$, (u, v) is an edge of $\overline{G}$ if and only if it is not an edge of G.

Why are perfect graphs interesting? A clique of size $k + 1$ is an easy certificate that a graph is not k-colorable, thus these graphs are exactly the graphs such that, as we color, we will always have a certificate that our coloring is optimal for the portion of the graph we have colored so far. We will see in the next talk that these graphs also have a nice characterization in terms of polytopes.

One could also view this class of graphs as an analog of bipartite graphs and Kőnig's Theorem. In general, the sizes maximum matchings and minimum vertex covers may differ, but Kőnig's Theorem says in bipartite graphs they coincide. By definition, perfect graphs are the graphs such that for every induced subgraph, the minimum number of colors in a proper coloring and the maximum size of a clique coincide. It turns out perfect graphs (and the weak perfect graph theorem) unify a lot of min-max results from early graph theory. There is a discussion of some of these results in one of Schrijver's textbooks [4]; we won't have time to discuss those results here.

To get some intuition for this class of graphs, let's classify some of our favorite graphs by whether they are perfect or not.

1.1 Some Perfect Graphs

Some classes of perfect graphs:

1. K_n (every induced subgraph is a k-clique and k-colorable.)
2. bipartite graphs (somewhat trivially)
3. cobipartite graphs, i.e. graphs G such that $\overline{G}$ is bipartite. (Corollary of Lemma 2)

Let's prove that last class of graphs is actually perfect. We need two tools to get there. The first is a simple relationship between colorings and cliques.

Lemma 1. *Let G be a graph, if G contains a clique of size k and a valid coloring using k colors, then $\omega(G) = \chi(G) = k$*

Proof. By finding a valid coloring and a clique of size k we have $\chi(G) \leq k$ and $\omega(G) \geq k$ but we always have $\omega(G) \leq \chi(G)$ (as we must use a new color for each vertex in the largest clique). Combining these inequalities gives $k \leq \omega(G) \leq \chi(G) \leq k$. □

The second tool in showing cobipartite graphs are perfect is Hall's Condition for the existence of matchings in bipartite graphs. We say a matching "saturates" a set of vertices if every vertex in that set has an incident edge in the matching.

Theorem 1 (Hall's Condition)**.** *Let G be a bipartite graph with partite sets S, T. There is a matching saturating S if and only if for all $S' \subseteq S$, $|N(S')| \geq |S'|$.*

For a subset $S' \subseteq S$, we define the **deficiency** of S' to be $|S'| - |N(S')|$. A set has positive deficiency if and only if it violates Hall's Condition.

With our new definition and facts in hand we can move to proving that cobipartite graphs are perfect.

Lemma 2. *Every cobipartite graph G satisfies $\chi(G) = \omega(G)$.*

Proof. Let S, T be a bipartition of $\overline{G}$, where $|S| \geq |T|$. We consider two cases:

Case 1: There is a matching saturating T in $\overline{G}$.
In G, S forms a clique. Coloring G with $|S|$ colors will complete the argument (by Lemma 1). Give a distinct color to every element of S, arbitrarily pick a matching of $\overline{G}$ saturating T, and give $t \in T$ the color of its matched partner in S. By definition of complement, this is a valid coloring.

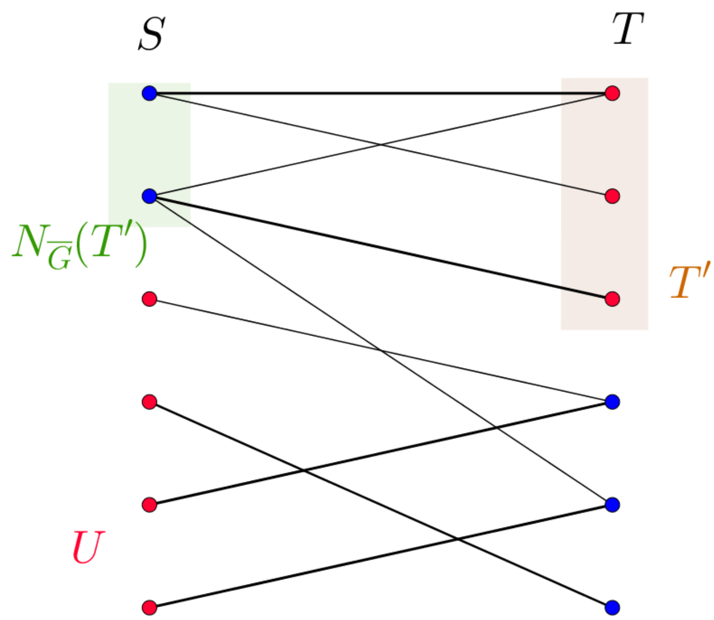

Figure 1: A sample $\overline{G}$, with extremal T'. The vertices of U are shown in red, and the matching of $\overline{G}$ used to create the coloring is in bold.

Case 2: No matching saturates T in $\overline{G}$.
By Hall's Condition, there is a set $T' \subseteq T$ such that $N_{\overline{G}}(T')$ is smaller than T'. Choose T' to have the maximum deficiency (in $\overline{G}$) among all such subsets. Now consider $U = (S \backslash N_{\overline{G}}(T')) \cup T'$. Note that U forms a clique in G (there are no edges in $\overline{G}$ among these edges by the bipartition and the fact that we removed $N_{\overline{G}}(T')$). See Figure 1. As before, it suffices to color G with $|U|$ colors. Assign a distinct color to each element of U. We claim there is a matching in $\overline{G}$ between $U \cap S$ and $T \backslash T'$, saturating $T \backslash T'$; if there were not, Hall's condition would give us another set T'' with positive deficiency, but then $T' \cup T''$ would violate the extremality of T'. We color $T \backslash T'$ the same as its partner in the matching. Similarly, we can find a matching in $\overline{G}$ between T' and $S \backslash U$ saturating $S \backslash U$. Indeed, if there were not such a matching then there would be a set $S' \subseteq N_{\overline{G}}(T')$ such that $|S'| > |N_{\overline{G}}(S')|$. But now $T' \backslash N_{\overline{G}}(S')$ has neighborhood of size at least $|N_{\overline{G}}(T') \backslash S'|$, which would force $T' \backslash N_{\overline{G}}(S')$ to have a larger deficiency than T', a contradiction. It is easy to check our coloring is valid – we use each color at most twice, and when we repeat a color it is assigned to a partner in a matching in $\overline{G}$, thus no edge in G is monochromatic, and $\chi(G) = \omega(G)$, as required. $\square$

1.2 Some Imperfect Graphs

Not all graphs are perfect. Here are some *Non-examples:*

1. $C_{2k+1} (k \geq 2)$ (which are not 2-colorable, but have no K_3).
2. $\overline{C_{2k+1}} (k \geq 2)$. (The complement of a clique is an independent set. The largest independent set in C_{2k+1} is size k; the complement of a coloring is a partition into cliques, the only cliques in cycles are edges, so we require $k + 1$ colors.)

2 The Weak Perfect Graph Theorem

You might have noticed a pattern: when a graph is perfect, so is its complement (and conversely). This is not a coincidence, in fact it's the Weak Perfect Graph Theorem.

Theorem 2 (Weak Perfect Graph Theorem [3])**.** *G is perfect if and only if $\overline{G}$ is perfect.*

The Weak Perfect Graph Theorem was first conjectured by Berge in 1960. By 1972, it was a well-known conjecture. In the same year that Lovasz resolved the conjecture in the affirmative, he produced a second proof, using different techniques that was simpler than the original [2].

Our first step is a strange-seeming Lemma about "duplicating" vertices. We say we duplicate a vertex x, by creating a new vertex x' such that $N(x') = N(x) \cup \{x\}$. We will call x' a *copy* of x. It turns out this operation preserves perfection:

Lemma 3. *If G is perfect and G' arises from G by duplicating a vertex, then G' is also perfect.*

Proof. We proceed by induction on the number of vertices in G. Our base case is trivial (a single vertex duplicates to an edge, both of which are complete graphs, and thus perfect). Now let G be a perfect graph and let G' arise from G by duplicating a vertex x, adding the vertex x'. We only need concern ourselves with whether $\chi(G') = \omega(G')$ – consider a proper induced subgraph H: if H contains at most one of x, x' it is isomorphic to an induced subgraph of G and its perfection follows from that of G. On the other hand, if H contains both x, x' but is a proper subgraph, it can be handled by the inductive hypothesis, so only G' itself remains. If $\omega(G') = \omega(G) + 1$, then we can take a coloring of G make x' a new color and be done by Lemma 1. Thus suppose $\omega(G') = \omega(G)$. We again color G' with $\omega(G')$ colors. Fix an optimal coloring of G. In this coloring, let x be colored blue and let X be the class of blue nodes. By the equality $\omega(G') = \omega(G)$, we know that x does not participate in any maximum cliques (otherwise adding x' to the clique would create one of size $\omega(G) + 1$).

Now consider the graph $H := G\backslash(X - x)$ i.e. G with all the blue nodes *except* x removed. Now any $\omega(G)$-sized clique in G must contain a blue vertex (to be validly-colored), but this vertex cannot be x by our previous observation, thus $\omega(H) < \omega(G)$. Now by perfection of G we have $\chi(H) = \omega(H)$. Fix such a coloring. We now color G' by taking this coloring of H and adding $X \cup \{x'\}$ as a single (new) color class. This gives an $\omega(H) + 1 \leq \omega(G) = \omega(G')$ coloring of G', which completes our proof. □

We can now turn to the proof of the Weak Perfect Graph Theorem.

Proof of Theorem 2. We again proceed by induction on the number of vertices in G. Our base case, K_1, is isomorphic to $\overline{K_1}$, so both are perfect. Now consider a perfect graph G (with at least 2 vertices). Let $\mathcal{K}$ be the set of all vertex sets underlying cliques of G. Let $\mathcal{A}$ be the set of all maximum independent sets of G. Note the lack of symmetry in these definitions: $\mathcal{K}$ contains all cliques, $\mathcal{A}$ contains only the *maximum* independent sets. By IH, we need only consider whether $\chi(\overline{G}) = \omega(\overline{G})$. It suffices to prove $\chi(\overline{G}) \leq \omega(\overline{G})$. Our approach is to find a set $K \in \mathcal{K}$ such that $K \cap A \neq \varnothing$ for all $A \in \mathcal{A}$. In this situation, we observe that removing K from $\overline{G}$ reduces the clique number:

$$\omega(\overline{G - K}) = \alpha(G - K) < \alpha(G) = \omega(\overline{G})$$

Then we have:

$$\chi(\overline{G}) \leq \chi(\overline{G - K}) + 1 = \omega(\overline{G - K}) + 1 \leq \omega(\overline{G})$$

Where the first inequality follows from coloring $\overline{G}$ using the coloring for $\overline{G - K}$, the equality is from IH, and the second inequality is due to every maximum independent set of G (i.e. every maximum clique of $\overline{G}$) intersecting K.

Thus it suffices to prove that such a K exists in every perfect graph. Suppose it does not, that is for every $K \in \mathcal{K}$ there is an $A_K \in \mathcal{A}$ such that $K \cap A_K = \varnothing$.

Our goal in deriving a contradiction will be to find a subgraph $\tilde{G}$ whose chromatic number is larger than its clique number, but the only real way we have to lower-bound the chromatic number is by $|\tilde{G}|/\alpha(\tilde{G})$ for the subgraph $\tilde{G}$ we choose. In general this bound isn't very tight, but perhaps we can use some extra structure to tighten it. In particular, suppose that the A_K we just defined were mutually disjoint. Indeed, if $\tilde{G}$ is just $G\left[\cup_{K\in\mathcal{K}} A_K\right]$ then in this case $|\tilde{G}| = \alpha(G) \cdot |\mathcal{K}|$ (Every A_K has size $\alpha(G)$, and we just assumed they are disjoint). We would also have $\chi(\tilde{G}) = |\mathcal{K}|$ (we can color by giving each A_K a distinct color, since the A_K are maximum independent sets of G [and thus $\tilde{G}$], we can do no better). Thus if the A_K are disjoint, our bound on $\chi(\tilde{G})$ will be tight, and we might have hope. What do we do when they are not disjoint? Make them disjoint! Duplicate each repeated vertex so that a distinct copy appears in exactly one A_K. As long as this duplication doesn't change $\mathcal{K}, \mathcal{A}$, or the perfection of G in the wrong way, we will be fine. Lemma 3 allows us to do exactly this.

Formally, for every vertex $x \in G$, replace x with a clique on $|\{K \in \mathcal{K} | x \in A_K\}|$ vertices. Let G' be the resulting graph. Note that this process can be achieved via repeated duplication[1], so by Lemma 3, G' is also perfect. Since every $K \in \mathcal{K}$ duplicated a vertex for every $x \in A_K$, G' has $|\mathcal{K}|\alpha(G)$ vertices. By perfection of G' we have $\chi(G') = \omega(G')$.

We calculate $\chi(G'), \omega(G')$ in the hope of deriving a contradiction. By construction, any maximal clique in G' consists of all duplicated vertices arising from some $X \in \mathcal{K}$. Now the size of this clique in G' is exactly the number of times we duplicated each of the vertices in X, i.e. $|\{(x, K) : K \in \mathcal{K}, x \in X, x \in A_K\}|$. This expression is equal to $\sum_{K \in \mathcal{K}} |X \cap A_K|$. X is a clique of G while each A_K is an independent set of G, so they intersect in at most one vertex each. Moreover, $X \cap A_X = \varnothing$, so we have that this set is of size at most $|\mathcal{K}| - 1$.

Now let's try to bound $\chi(G')$. The only good way we have to bound it is to use the fact that $\chi(G') \geq |G'|/\alpha(G')$ (because every color class is an independent set). Let us try to bound the chromatic number this way. Again by definition of the graph

$$|G'| = |\{(x, K) : x \in V(G), K \in \mathcal{K}, x \in A_K\}| = \sum_{K \in \mathcal{K}} |A_K| = \alpha(G)|\mathcal{K}|$$

We have $\alpha(G') \leq \alpha(G)$ (a vertex and its copy cannot be in an independent set in G', on the other hand, we might have deleted a vertex when creating G'). Thus

$$\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{|G'|}{\alpha(G)} = |\mathcal{K}|$$

Now we have $\chi(G') \geq |\mathcal{K}| > |\mathcal{K}| - 1 \geq \omega(G')$, which contradicts the perfection of G'. Thus there is always a $K \in \mathcal{K}$ such that K intersects each $A \in \mathcal{A}$, and our previous argument guarantees perfection of G. □

References

[1] Reinhard Diestel. Graph theory, 2000.

[2] László Lovász. A characterization of perfect graphs. *Journal of Combinatorial Theory, Series B*, 13(2):95–98, 1972.

[3] László Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2(3):253–267, 1972.

[4] Alexander Schrijver. *A course in combinatorial optimization.*

[5] Douglas Brent West. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[1] One might be concerned that some vertex, v, might not appear in any A_K. In this case, do not duplicate v, and then let G' be the induced subgraph on the duplicated vertices, excluding any such v. This graph is still perfect, as it is an induced subgraph of a perfect graph.