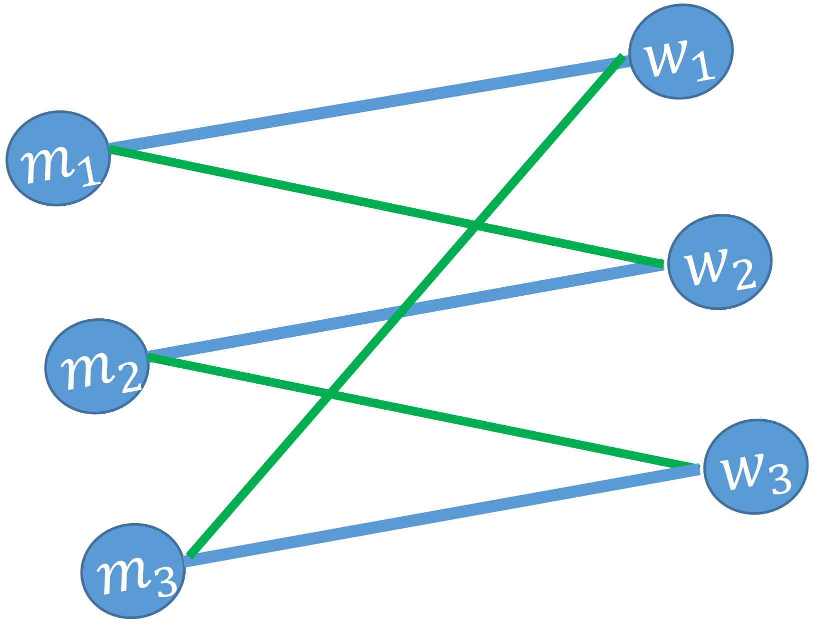
A Simply Exponential Upper Bound on the Maximum Number of Stable Matchings

Robbie Weber

Joint work with Anna Karlin and Shayan Oveis Gharan



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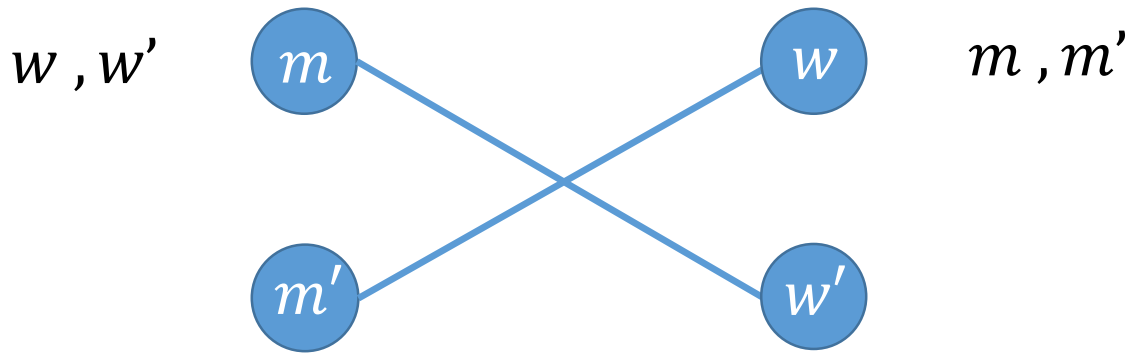

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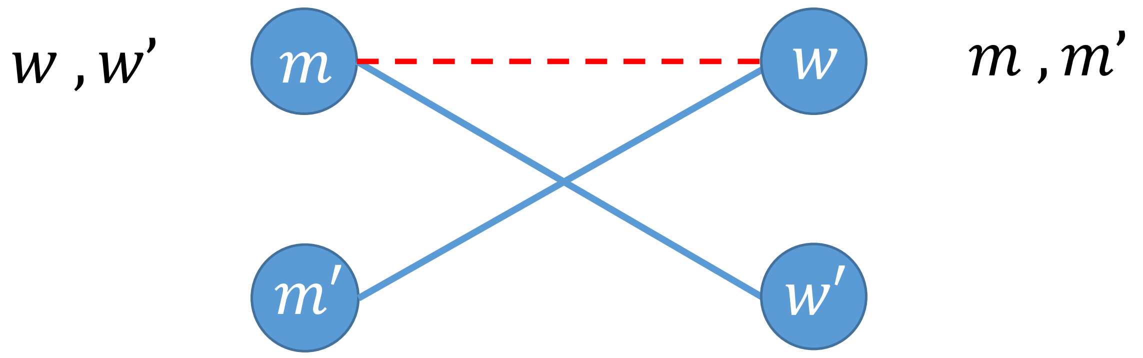

Stable Matching Problem

[Gale-Shapley ‘62]

There is always at least one stable matching,

and there can be more than one.

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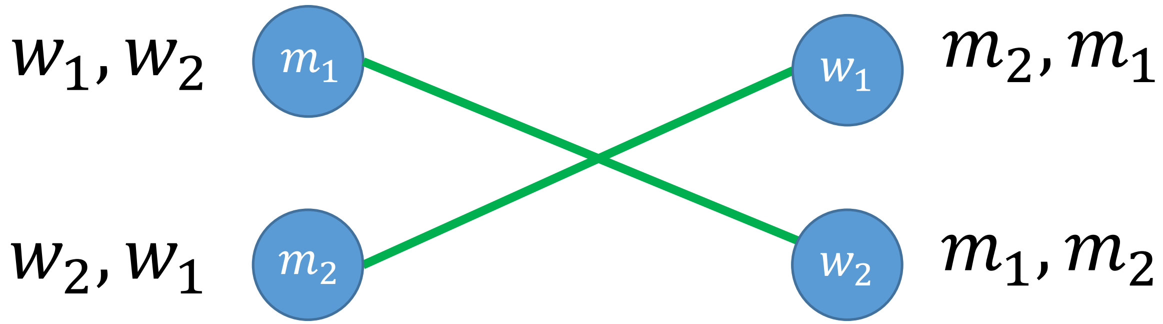

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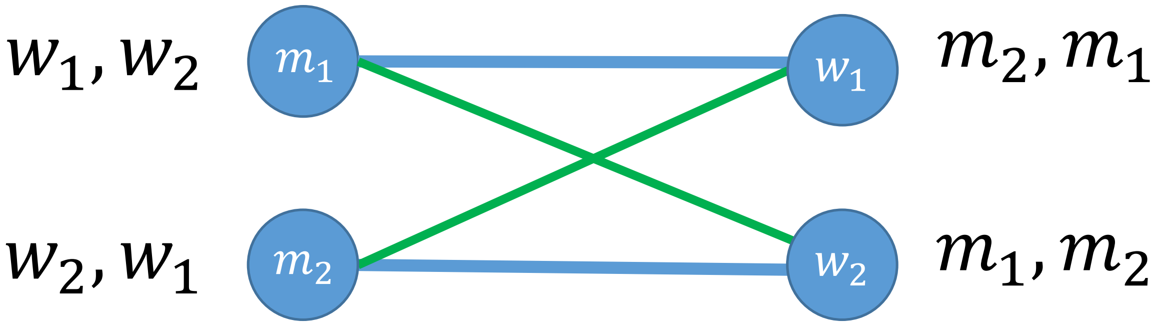

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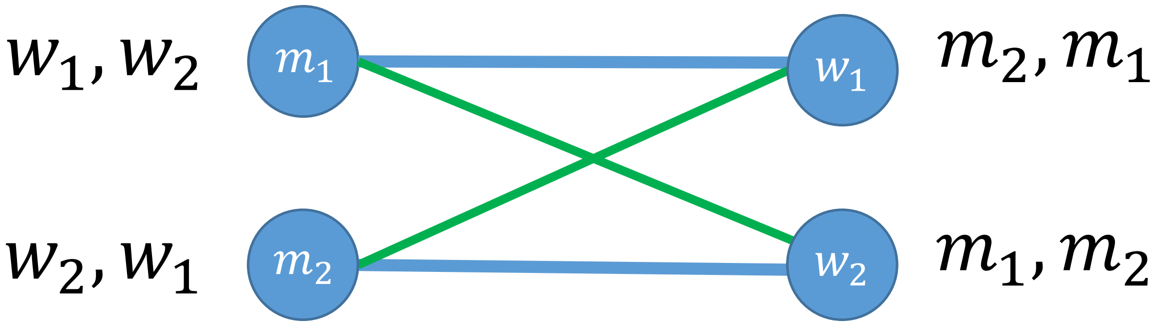

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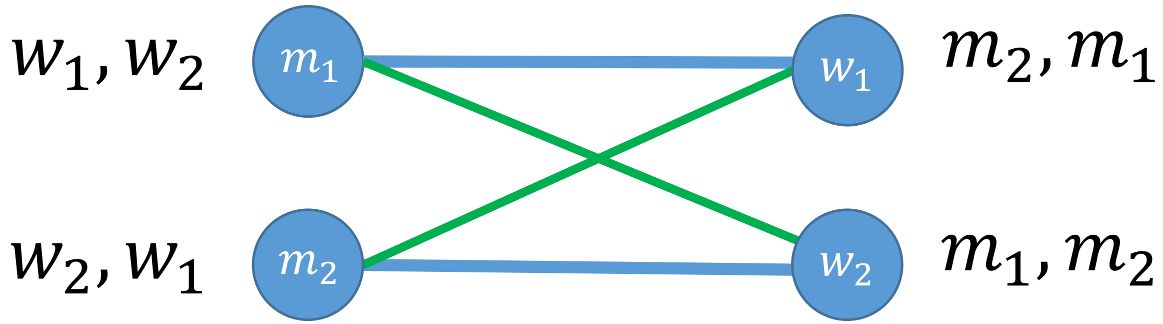

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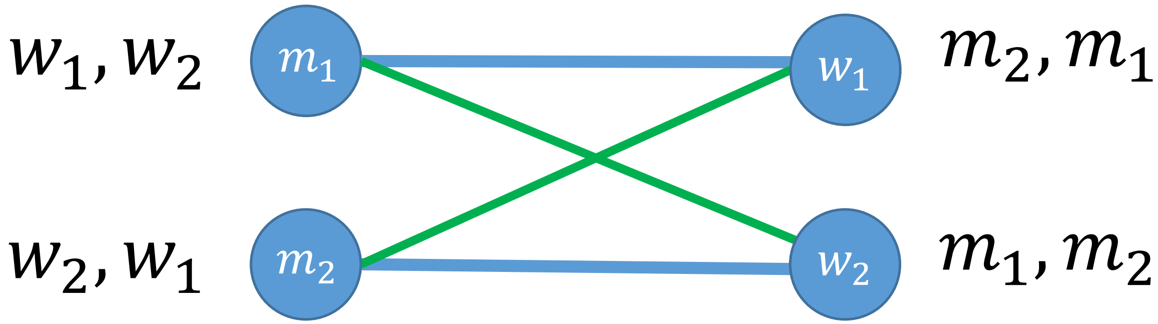

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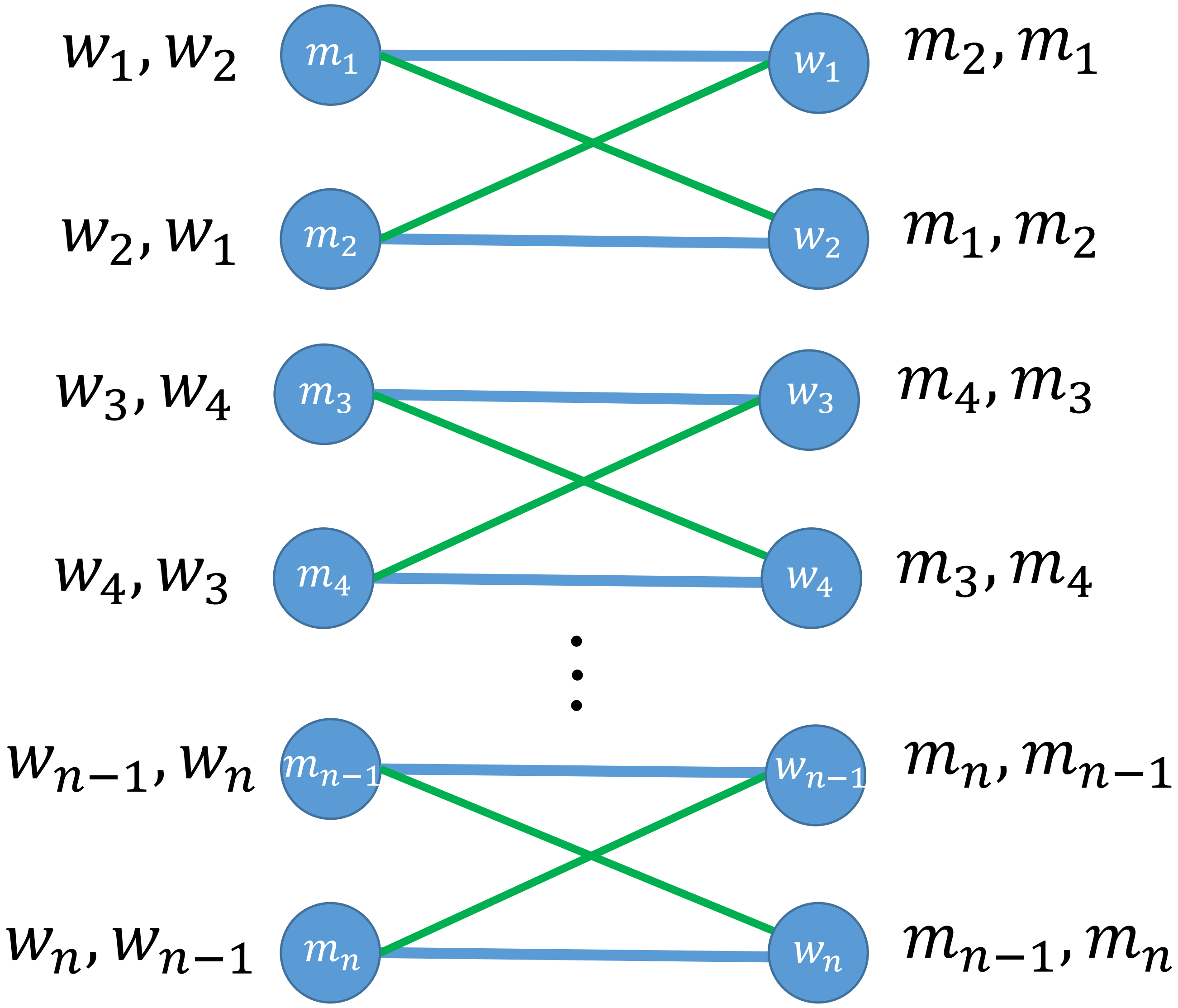

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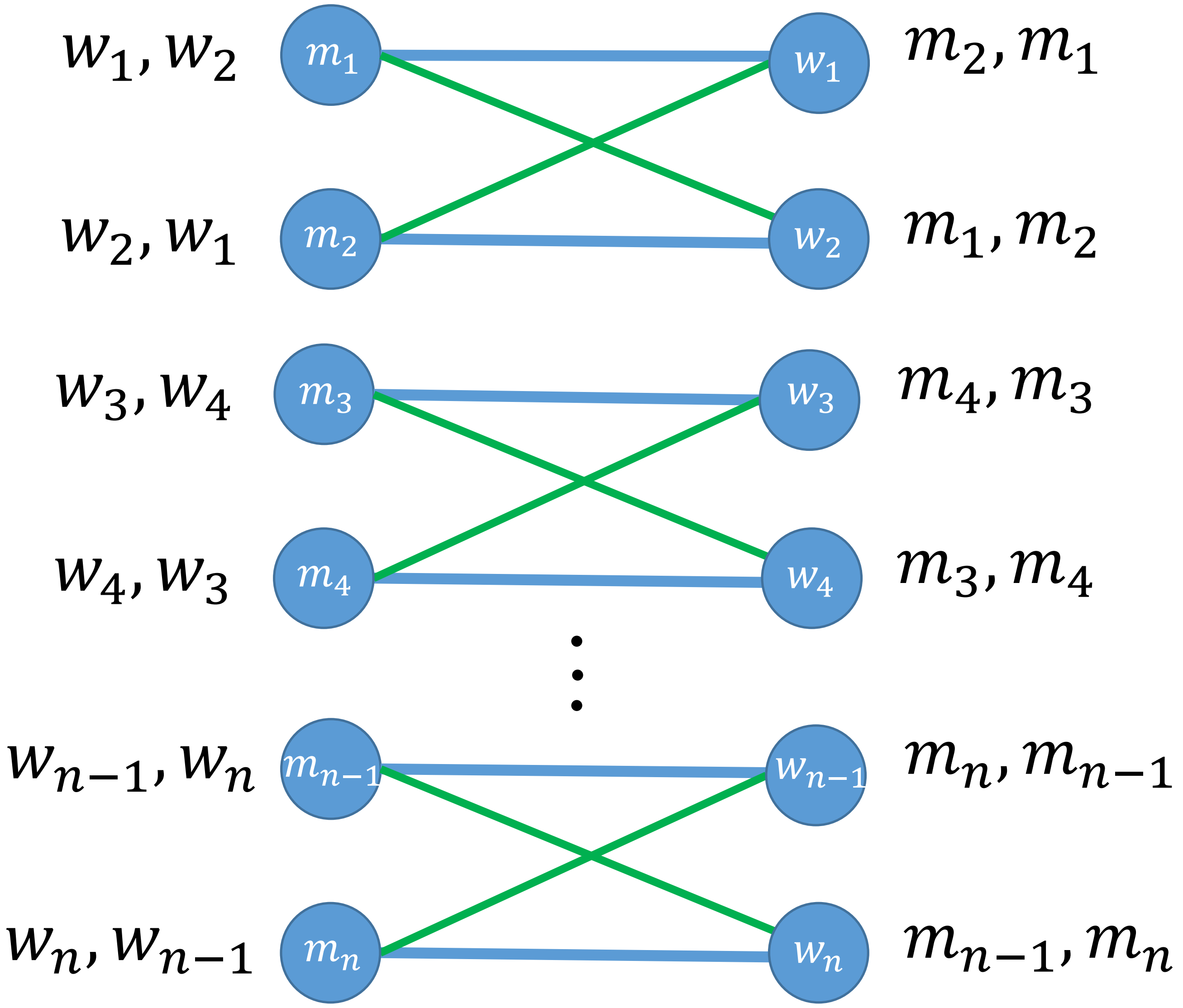
$\geq 2.28^n$ [Irving-Leather-Knuth’86, Benjamin-Converse-Krieger’95, Thurber’02]



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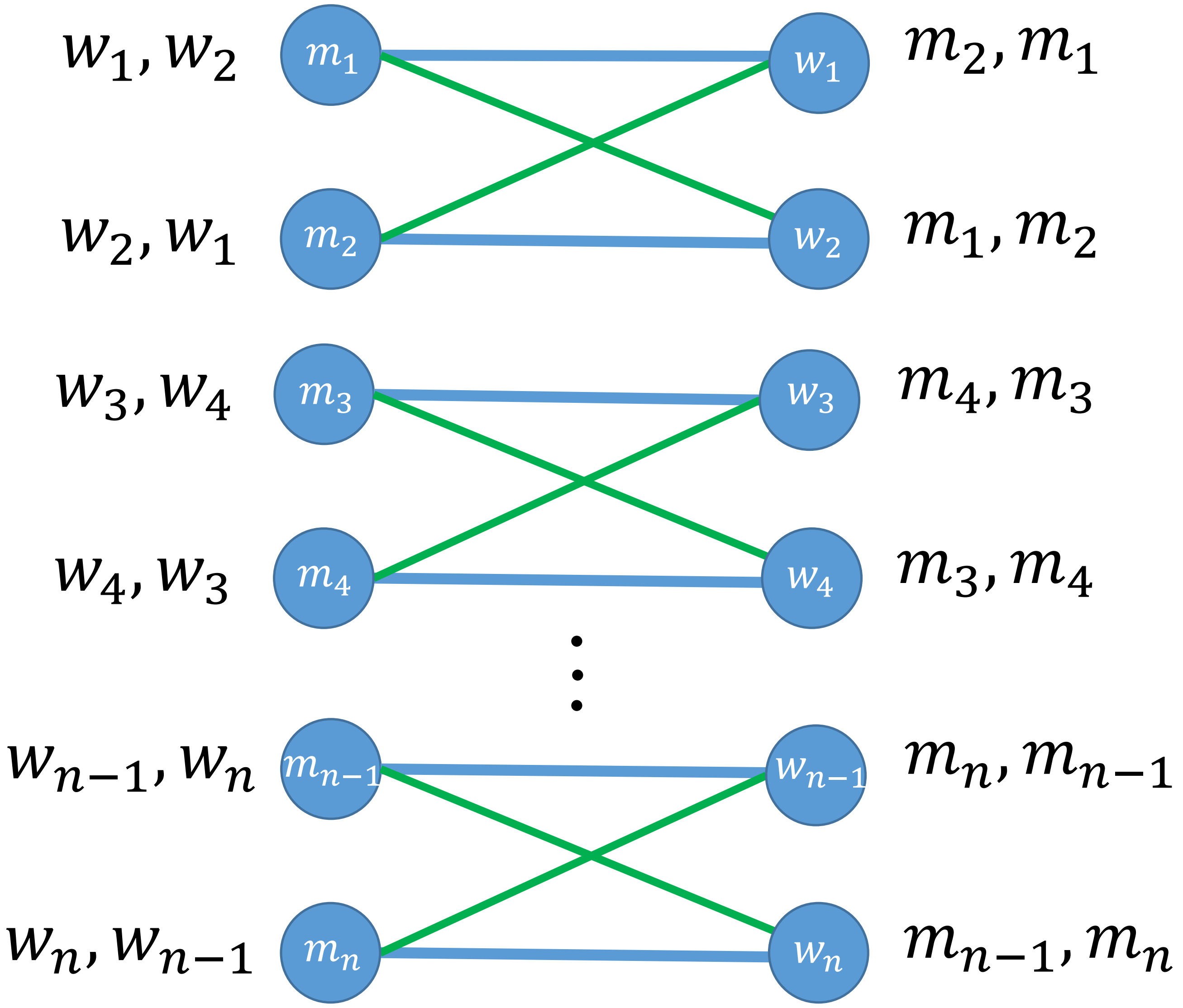

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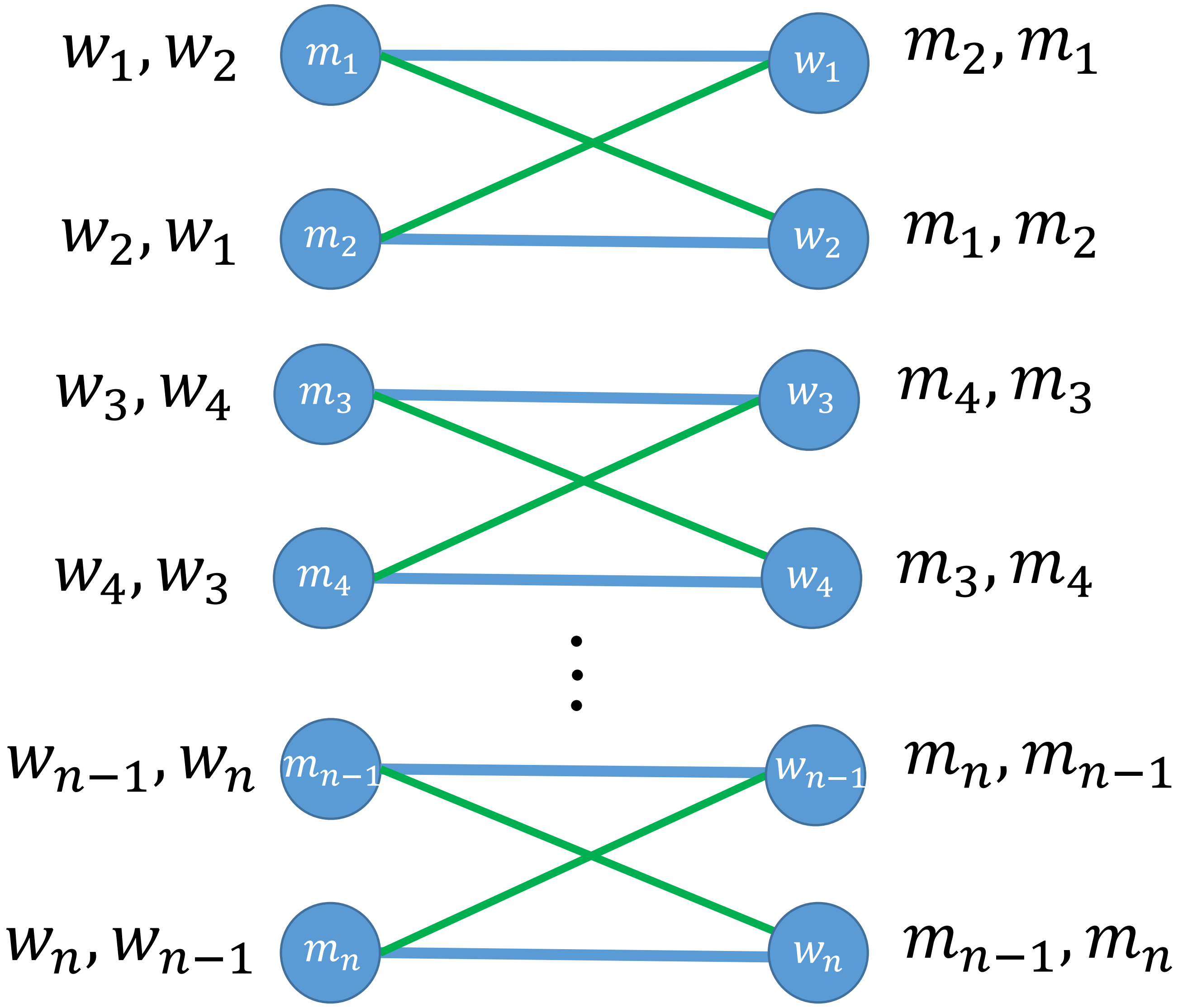
$\leq n!/c^n$ [Stathopoulos'11]



Main Result

There is a universal constant C such that every stable matching instance with n men and n women has $\leq C^n$ stable matchings.

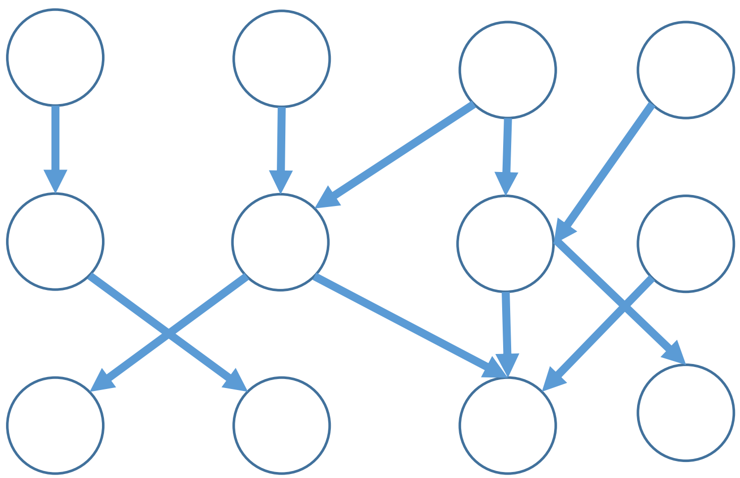
Proof Outline

Step 1 [Irving Leather 86]: The number of stable matchings equals the number of downsets of a certain POSET.

POSET: A set with a transitive antisymmetric relation $\prec$, i.e., a DAG

u dominates v if $v \prec u$.

Downset: a set of elements, and everything they dominate.



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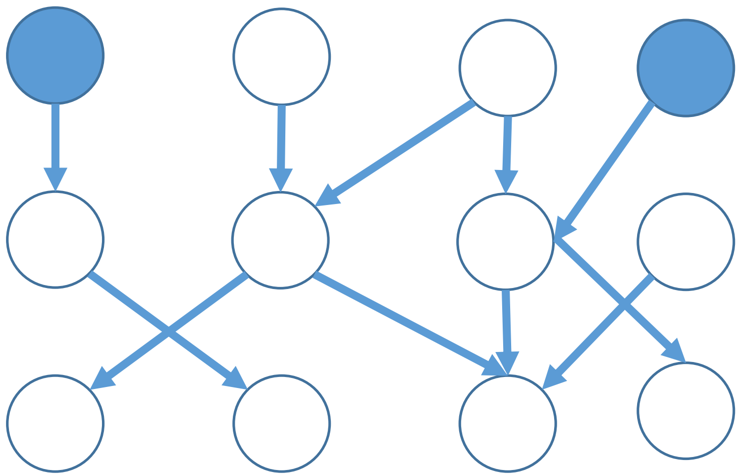

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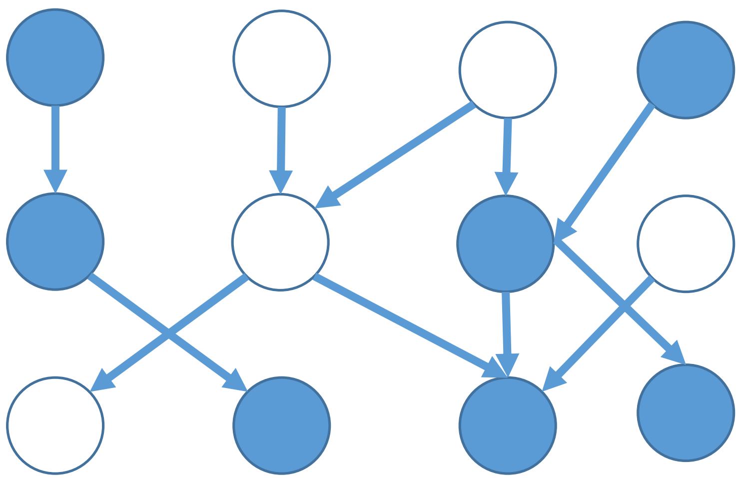

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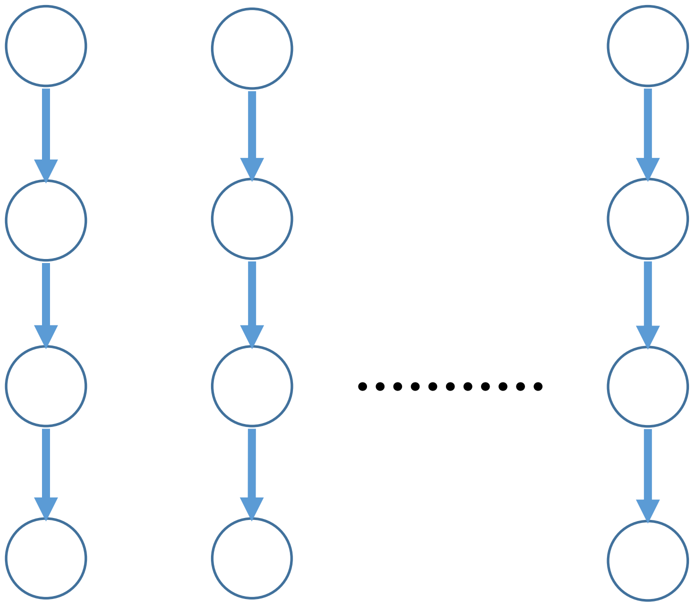

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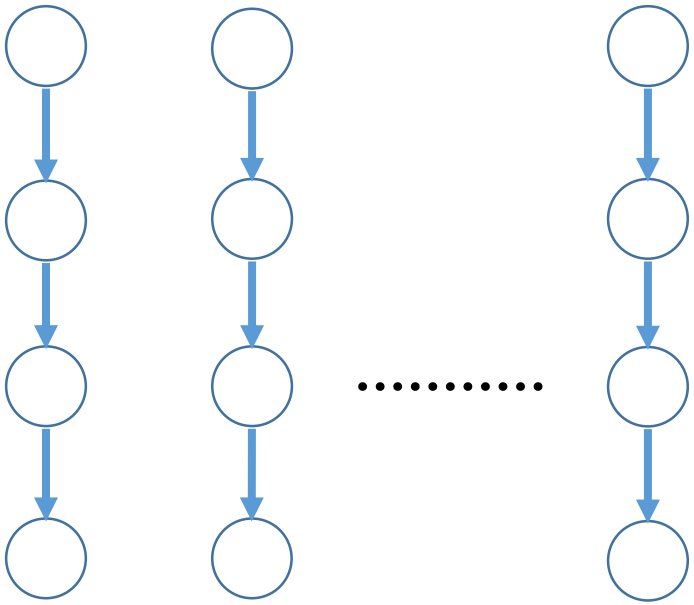

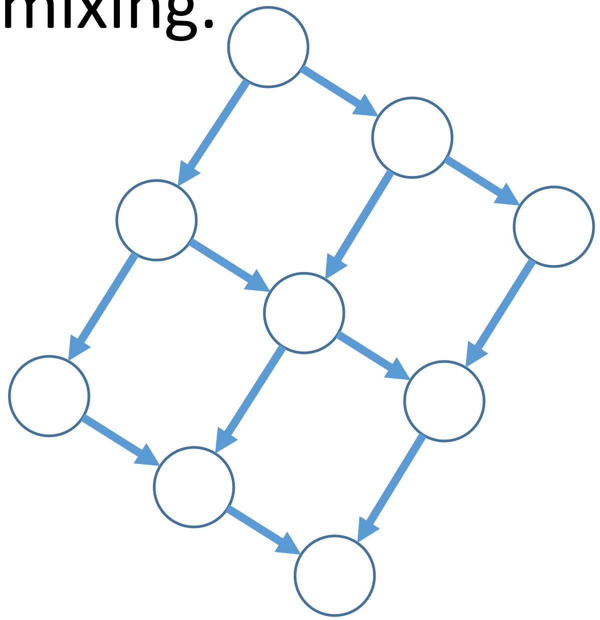

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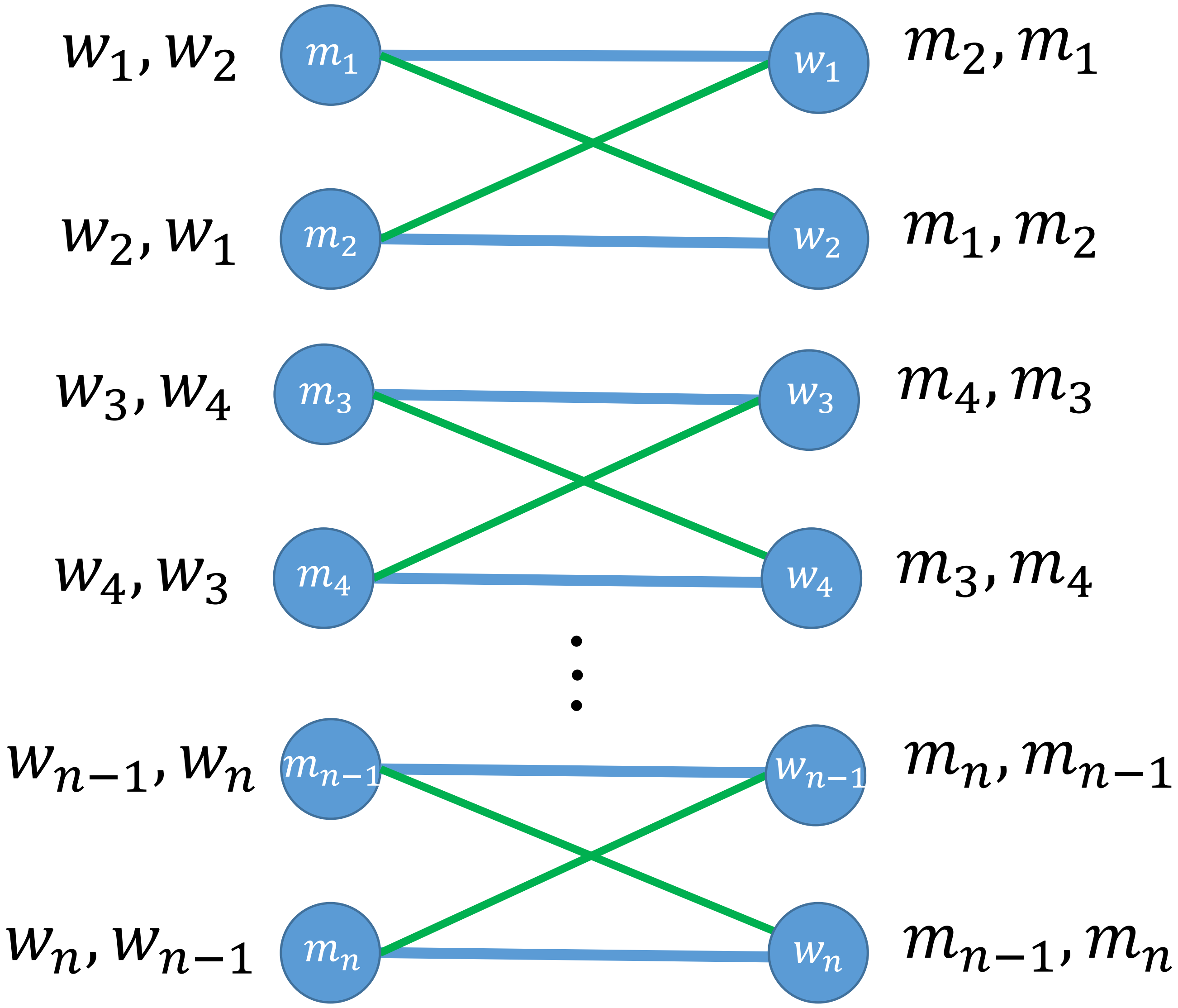
There can be exponentially many stable matchings

But there are only polynomially many “simple transformations” to jump from one stable matching to another.

Irving and Leather's POSET

A trivial example:



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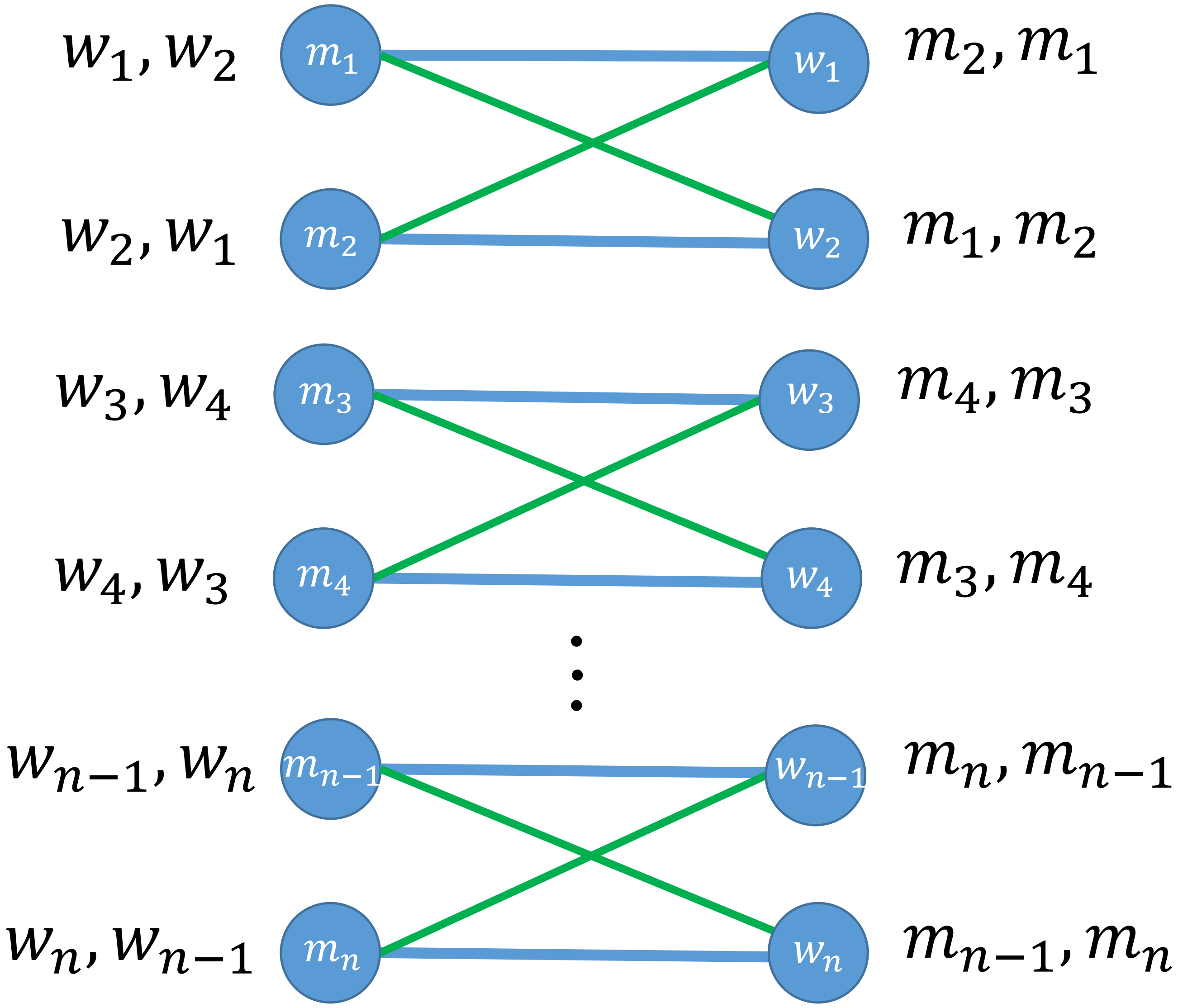

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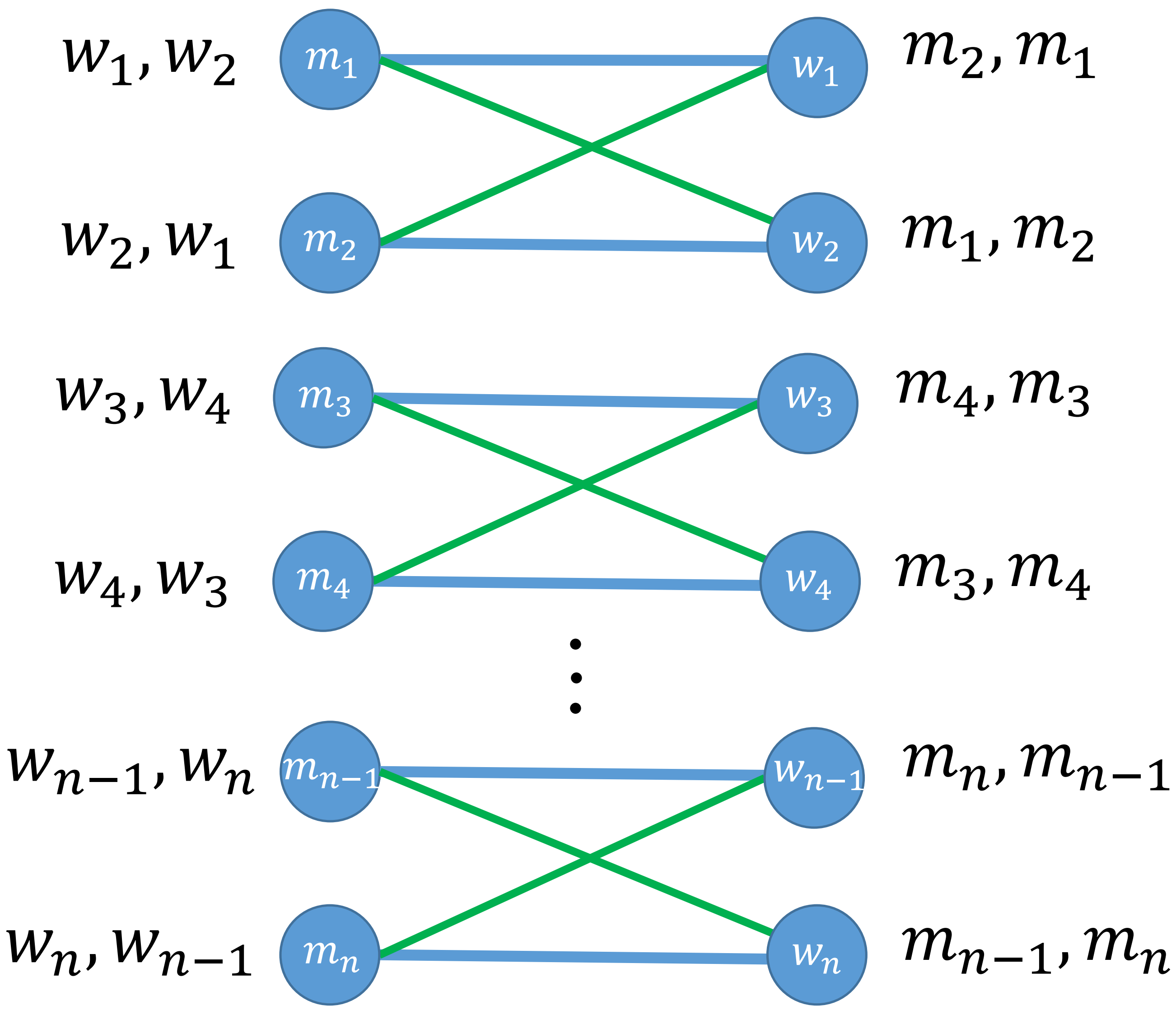
(Switching from the blue matching to the green matching in some pair)



Rotation

A rotation is an ordered list of pairs:

$$\rho = (m_1, w_1), (m_2, w_2), \ldots, (m_k, w_k)$$

s.t., for all stable matchings M where $\rho \subseteq M$

- $M' = M \setminus \rho \cup (m_1, w_2), (m_2, w_3), \ldots (m_k, w_1)$ is stable.
- Each m_i is worse off in M' than in M, and each w_i is better off.
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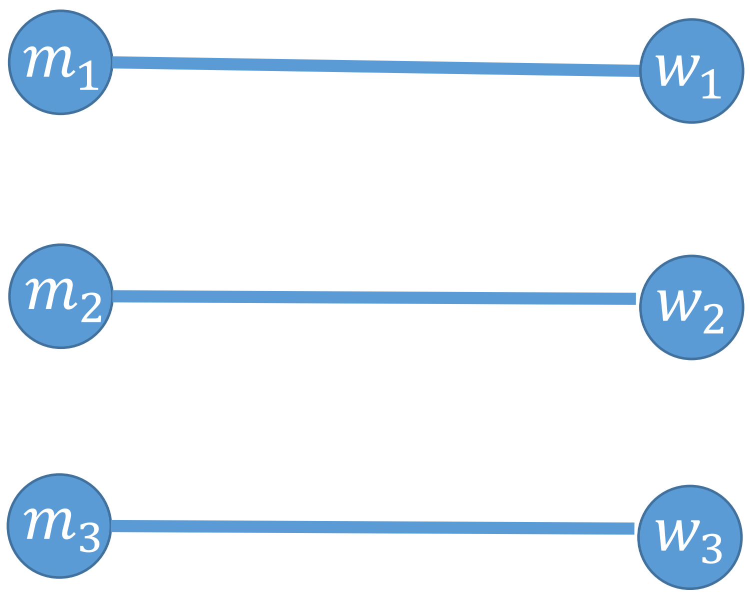

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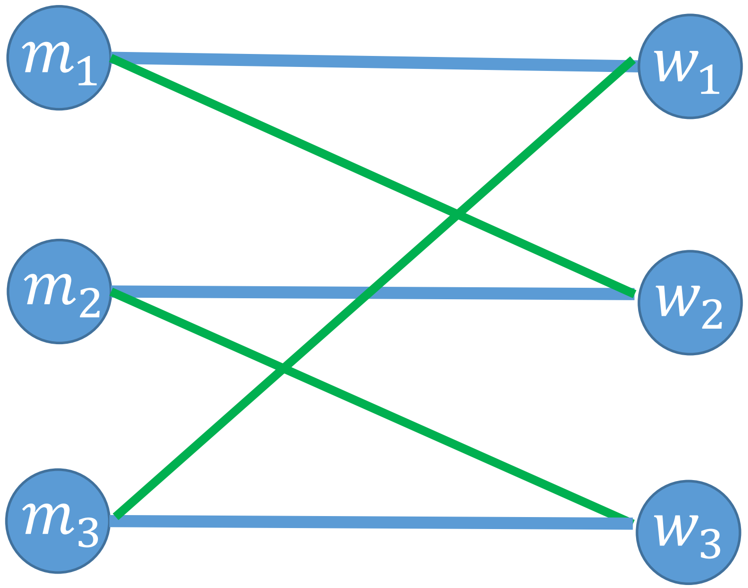

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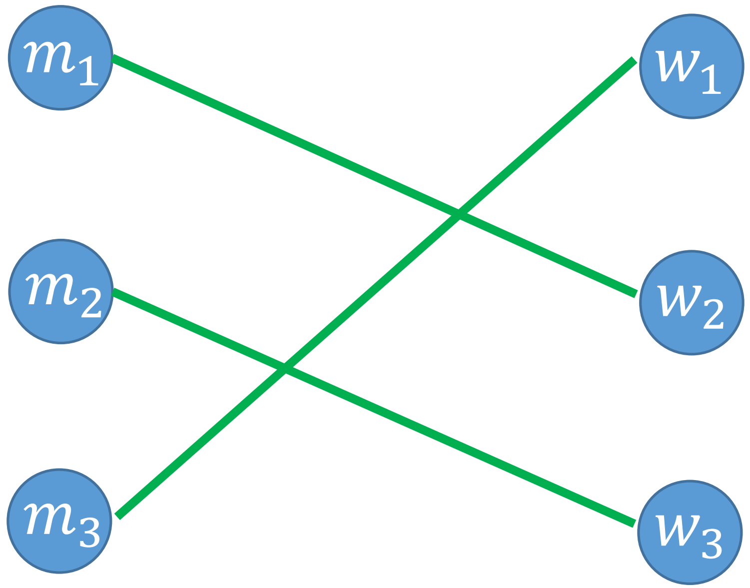
Any stable matching instance has at most $O(n^2)$ rotations that can be constructed in poly time.



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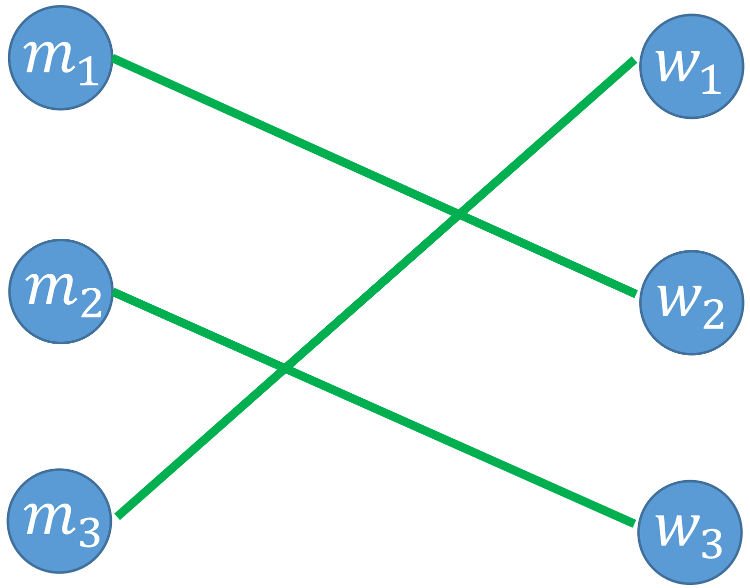

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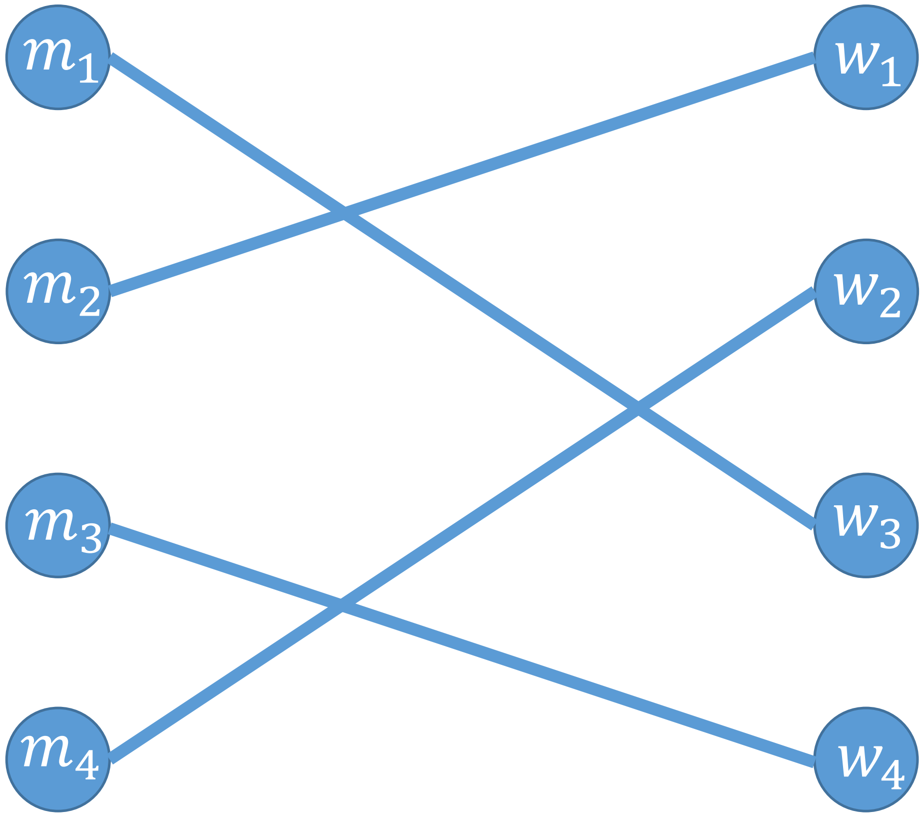
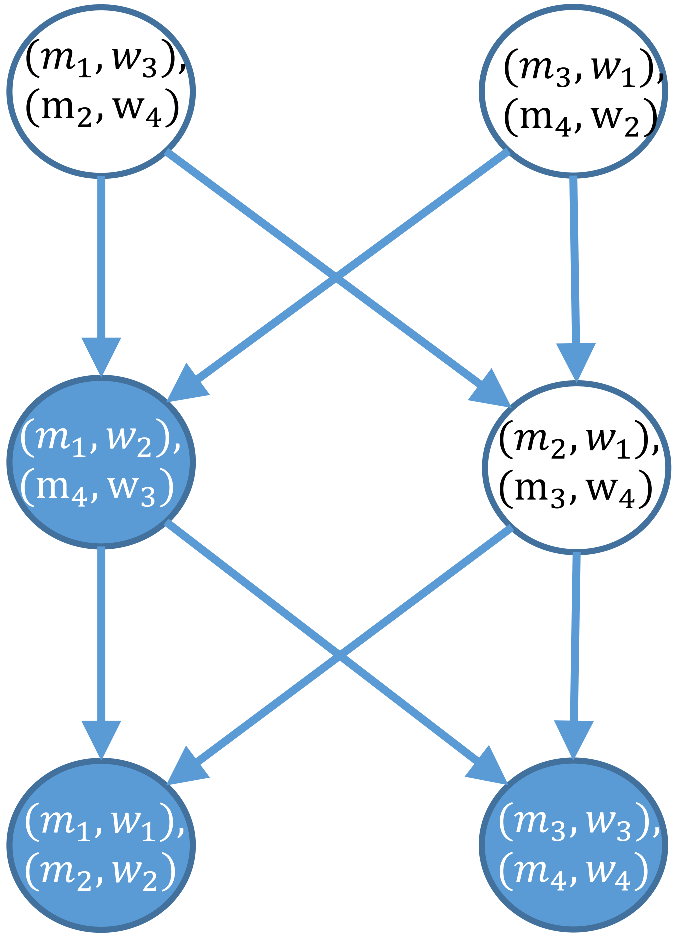
Observation: All rotations with an agent form a directed path.

[Irving-Leather 86]

#Stable Matchings = #Downsets Rotation POSET





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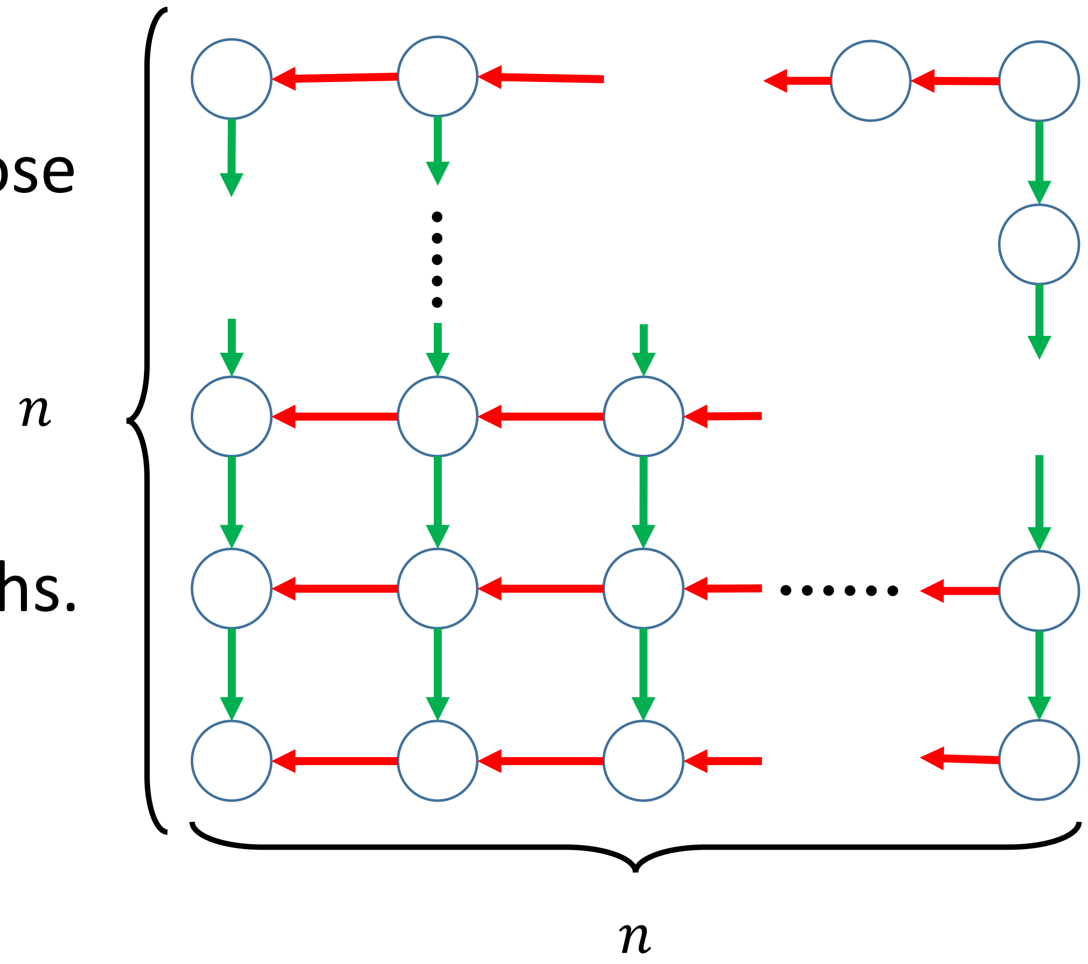
k-Mixing POSETs

A POSET is *k-mixing* if we can decompose it into k paths $P_1, P_2, \ldots, P_k$ s.t.,

- $\cup P_i$ includes all elements
- Every set U intersects $\geq \sqrt{|U|}$ paths.



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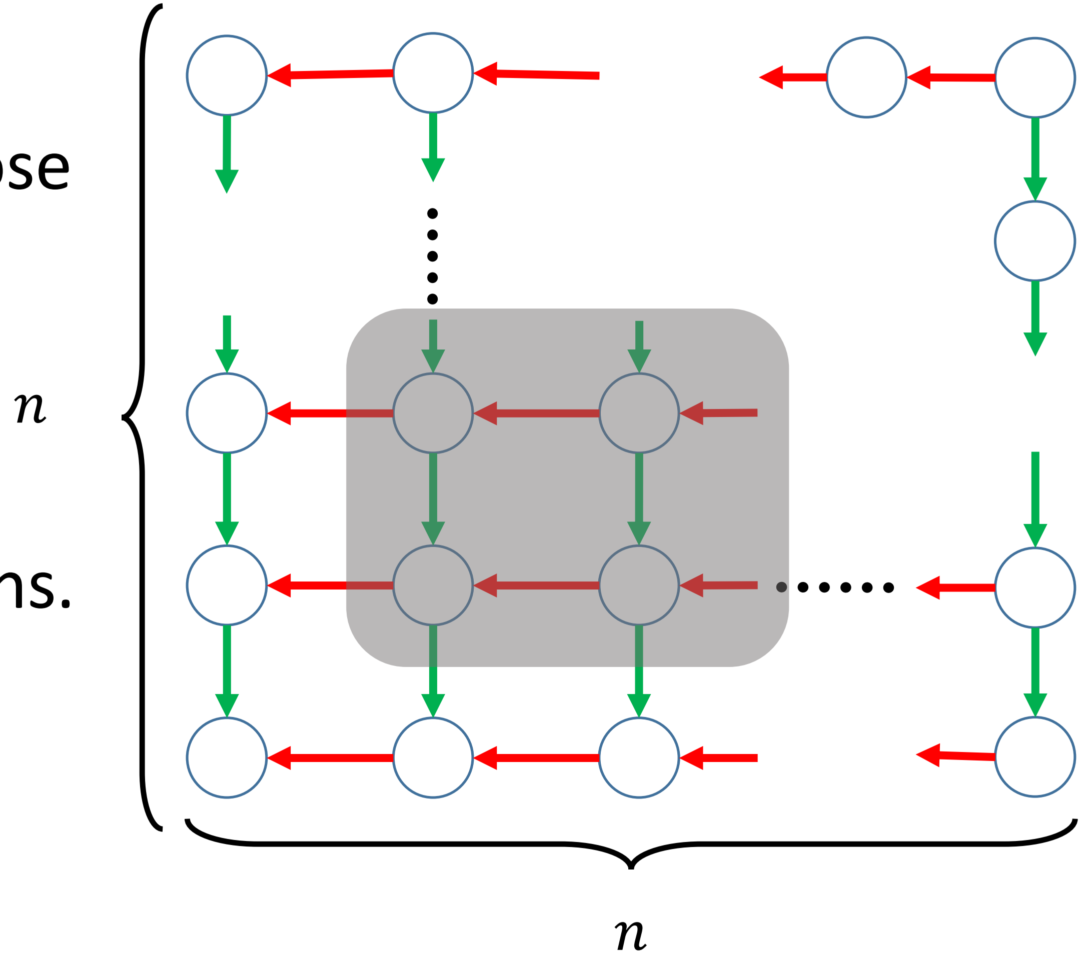

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There are $2n$ paths.

- These paths cover all rotations.

- They mix!
  1) Every rotation contains a new (man, woman) pair.
  2) We need at least $\sqrt{r}$ men and or women to make r new pairs. Thus r rotations must intersect at least $\sqrt{r}$ paths.

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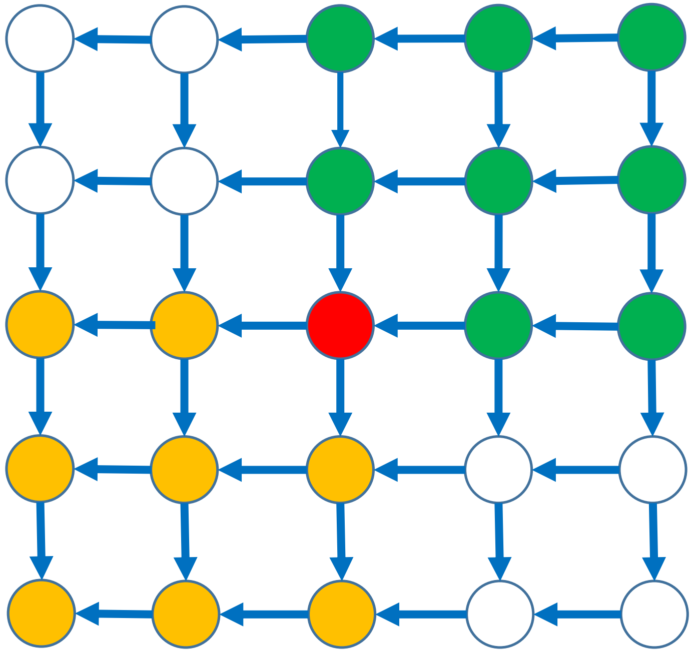
A node is α-critical if it dominates and is dominated by at least α nodes.

Fact: By inspecting an α-critical node, we can eliminate α nodes:



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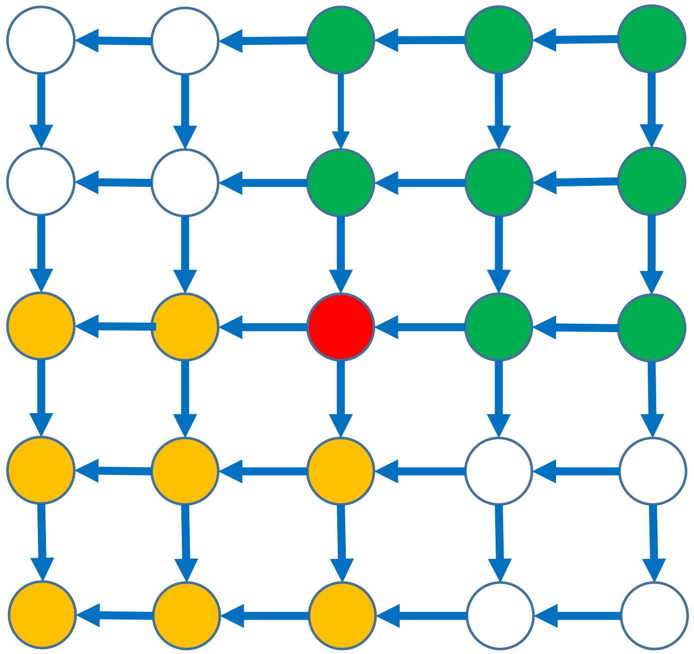

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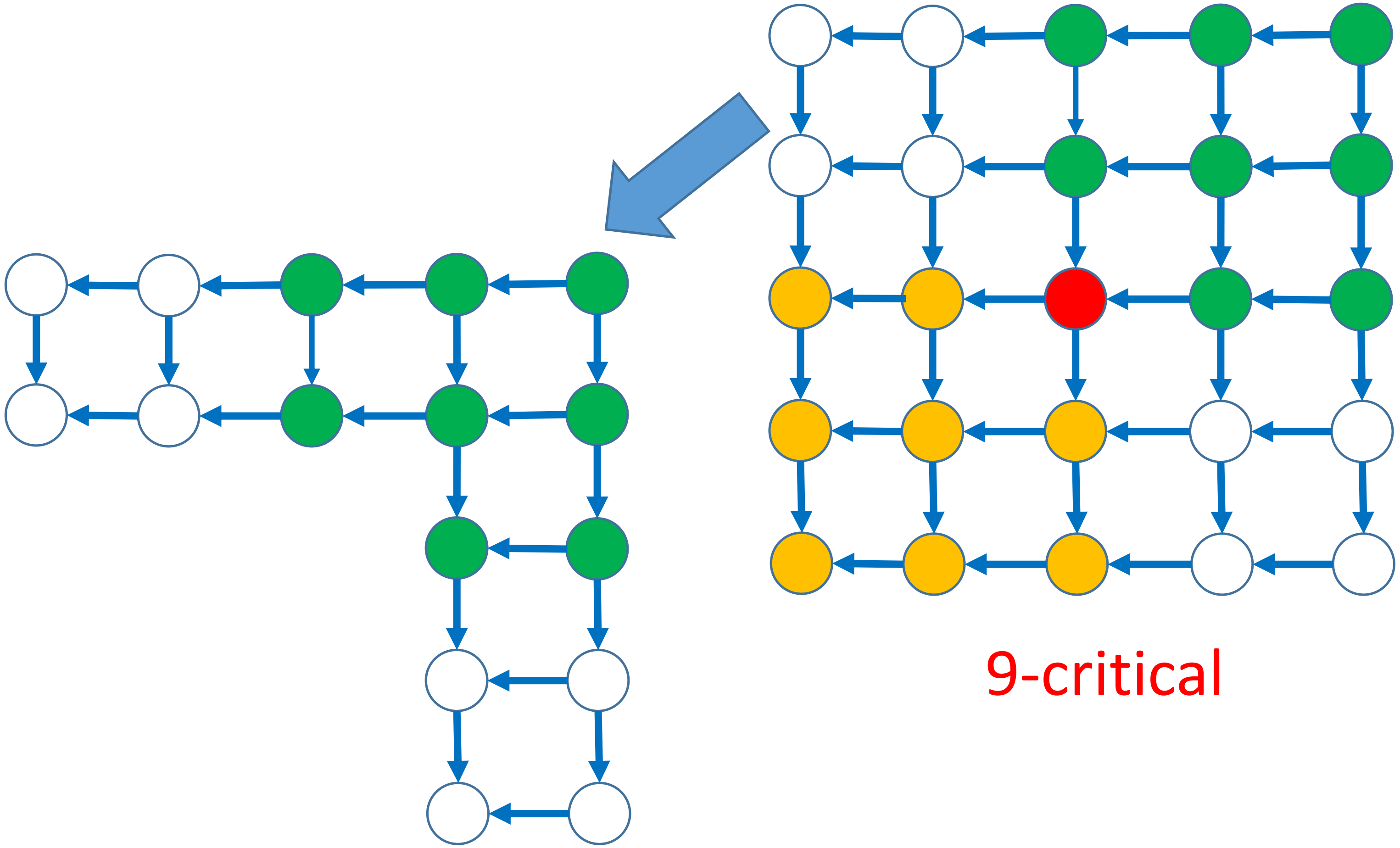

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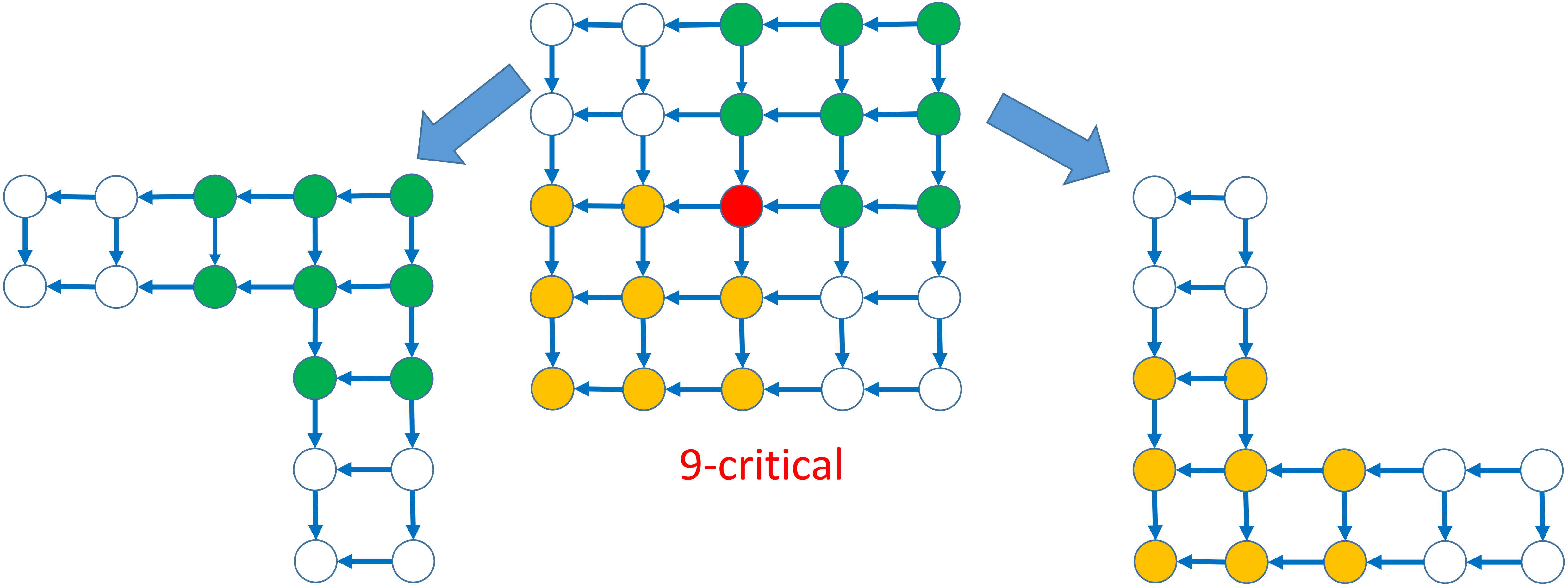

Critical Nodes ⇒ Main Theorem

$$\mathrm{T}(|\mathrm{V}|) \leq \begin{cases} 2\mathrm{T}(|V| - \alpha) & \text{if } |V| > k \\ 2^k & \text{if } |V| \leq k \end{cases}$$

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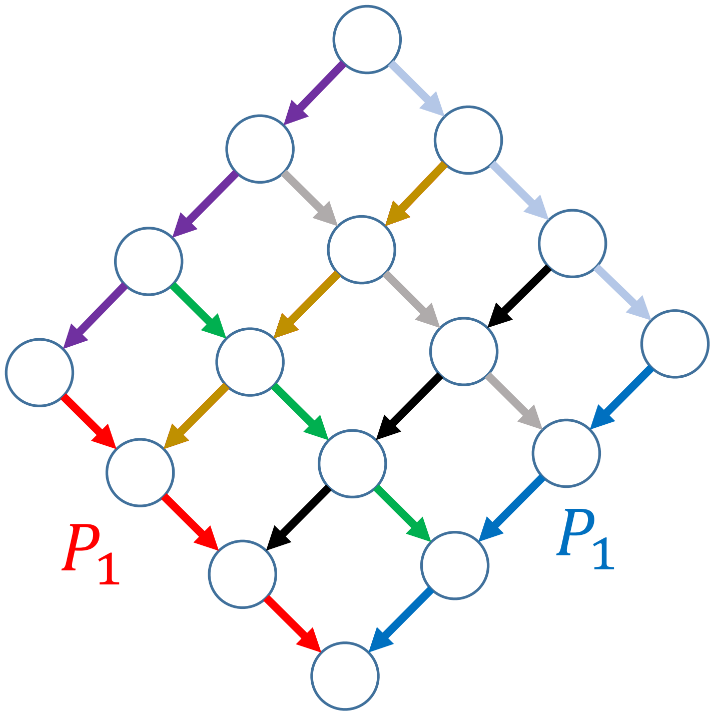
Unfolding the recurrence leads to a geometric series.

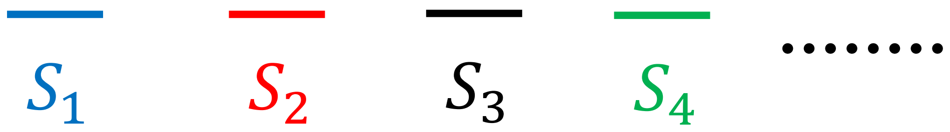
Which gives $T(|V|) \leq C^{|V|}$.

Constructing Disjoint Subpaths

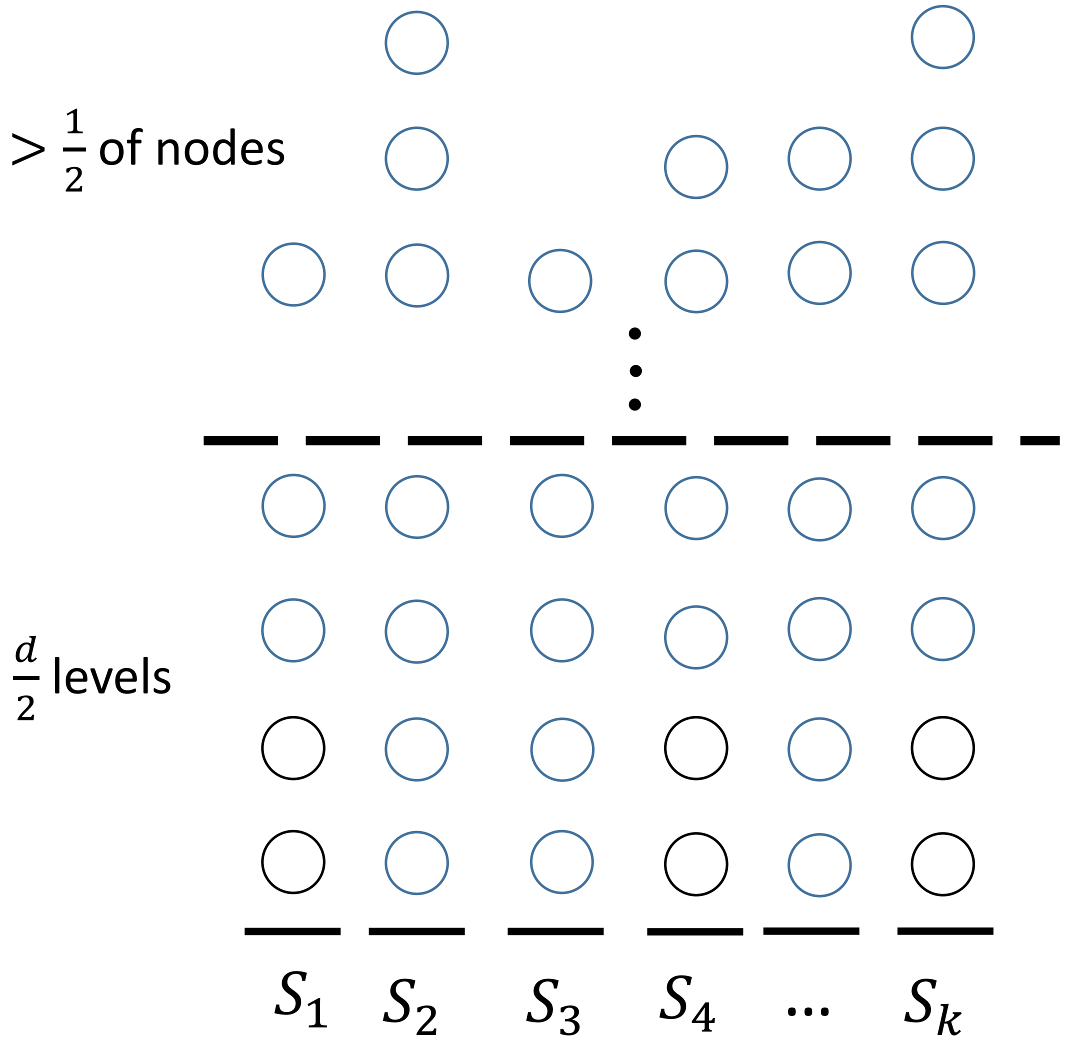
We construct a partition into subpaths $S_1, \ldots, S_k$ s.t.,

- Each $S_i \subseteq P_i$.
- If a node v dominates m nodes in its subpath S_i then it dominates m nodes in all S_j where $v \in P_j$.





Finding a half-Critical Node



k-mixing: Every set U intersects $\sqrt{|U|}$ of paths $P_1, \dots, P_k$

Partition: Define subpaths $S_1, S_2, \dots, S_k$ such that:

- Each $S_i \subseteq P_i$.
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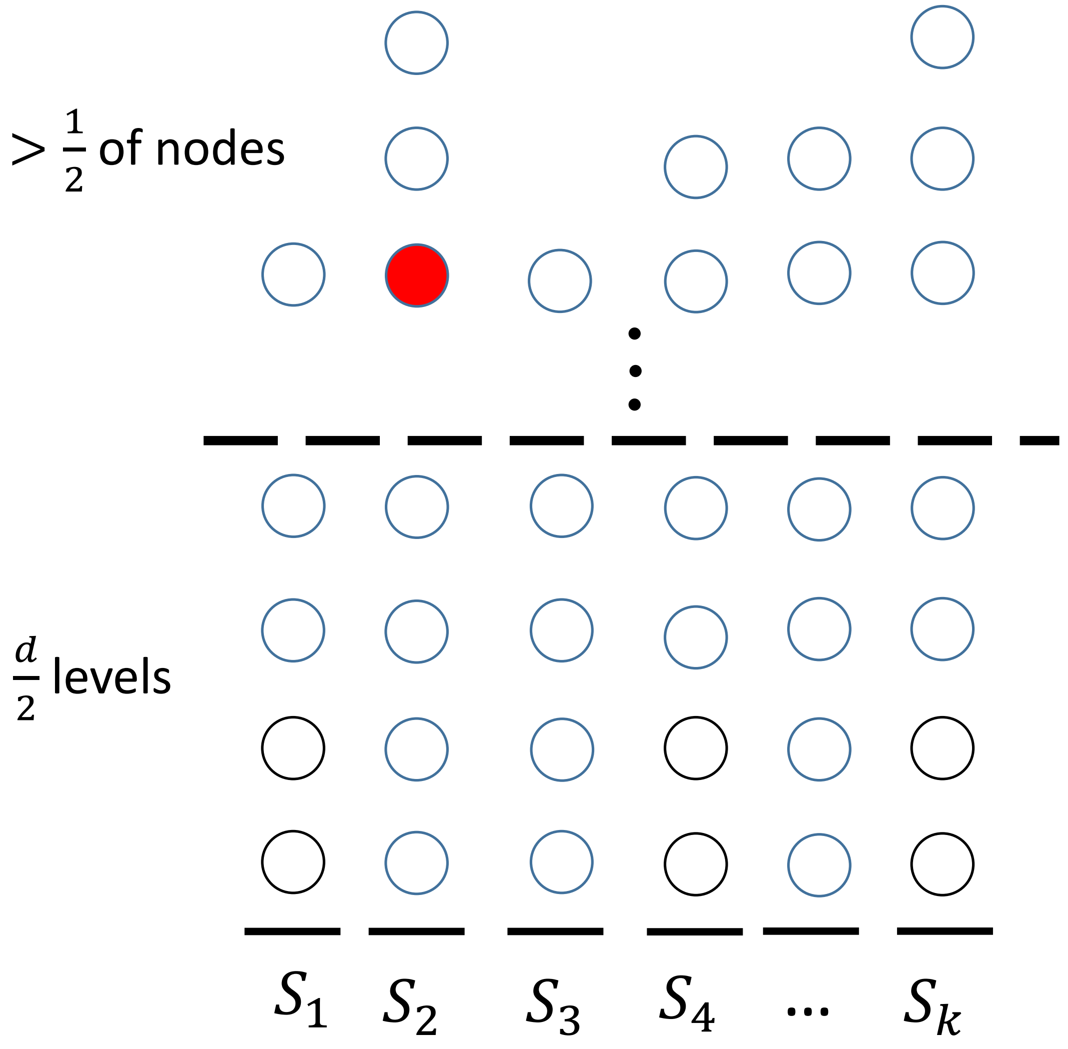
Claim: Most nodes dominates $\Omega\left(d^{3/2}\right)$ nodes.

Finding a half-Critical Node



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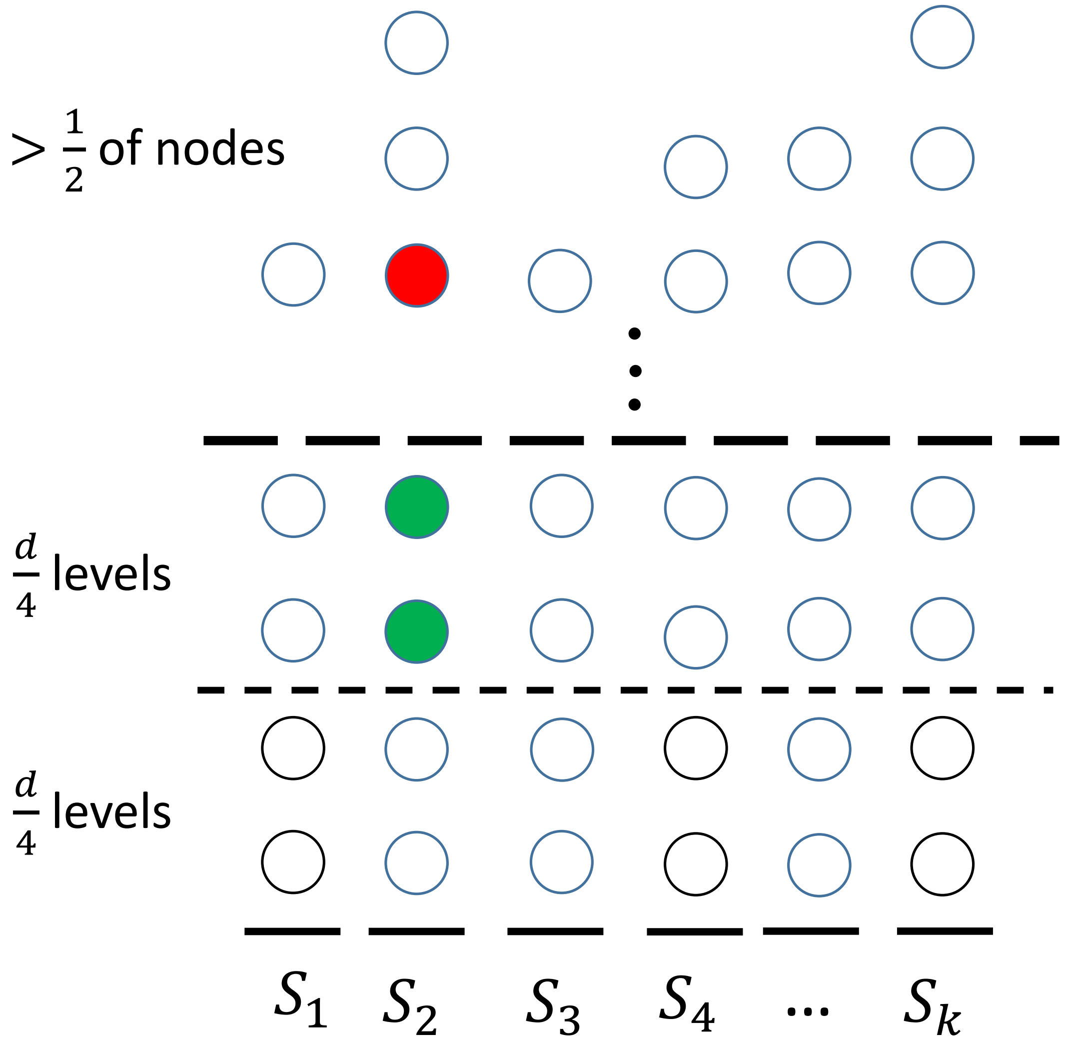

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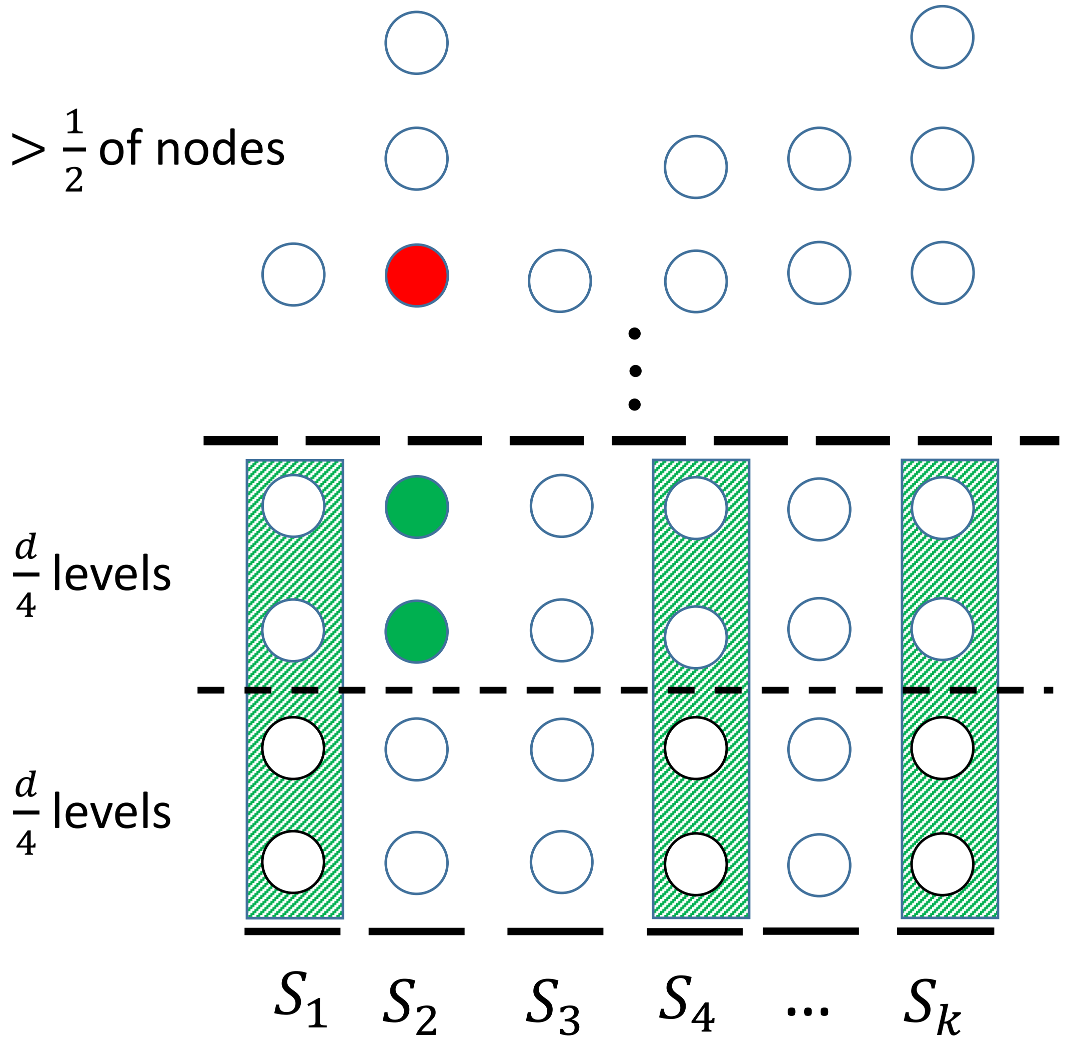
- Red node dominates $d/4$ green nodes

Finding a half-Critical Node



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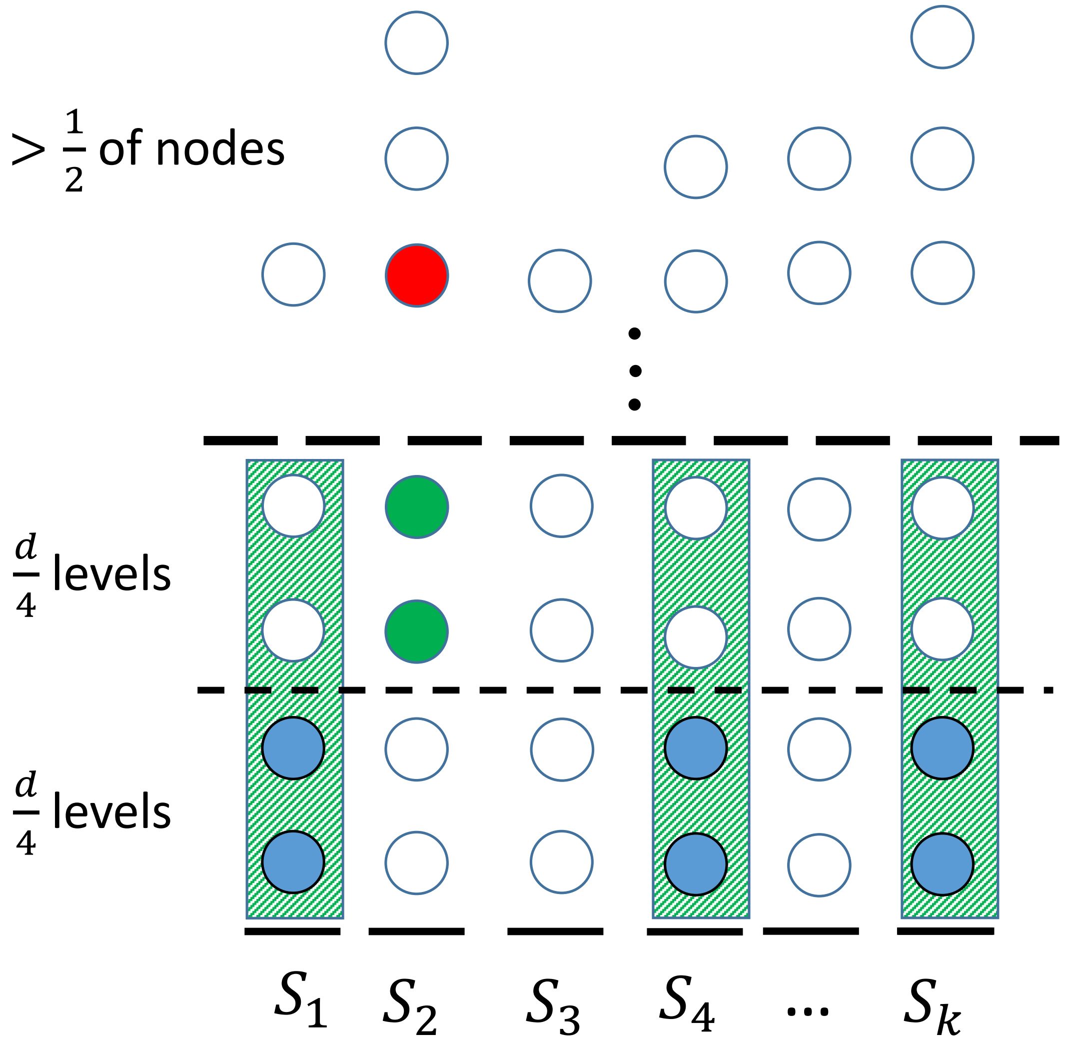
- Red node dominates $d/4$ green nodes
- Green nodes mix, so belong to $\Omega(\sqrt{d})$ paths.

Finding a half-Critical Node



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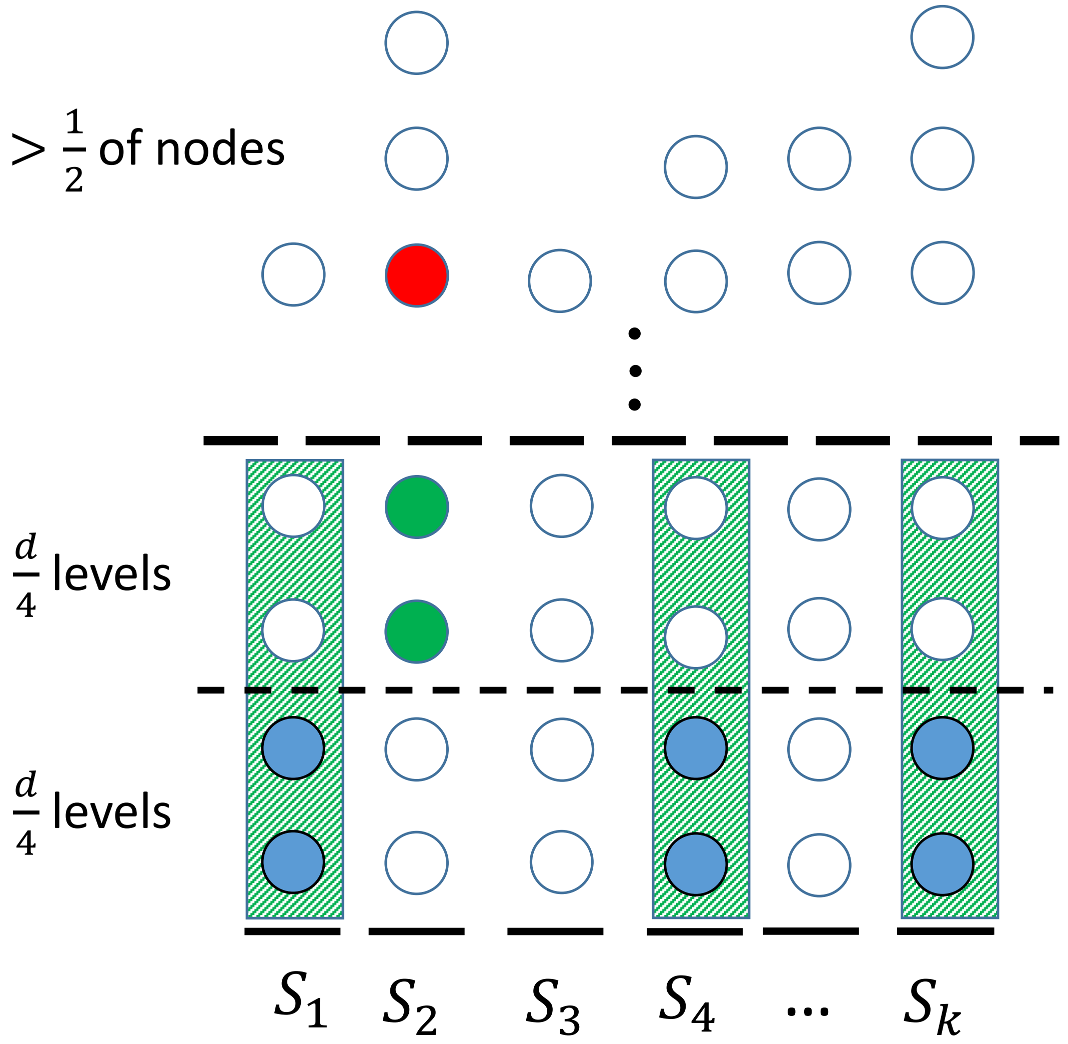
Claim: Most nodes dominates $\Omega\left(d^{3/2}\right)$ nodes.

- Red node dominates $d/4$ green nodes
- Green nodes mix, so belong to $\Omega(\sqrt{d})$ paths.
- Every green node dominates at least $d/4$ in every path that it appears.

Finding a half-Critical Node



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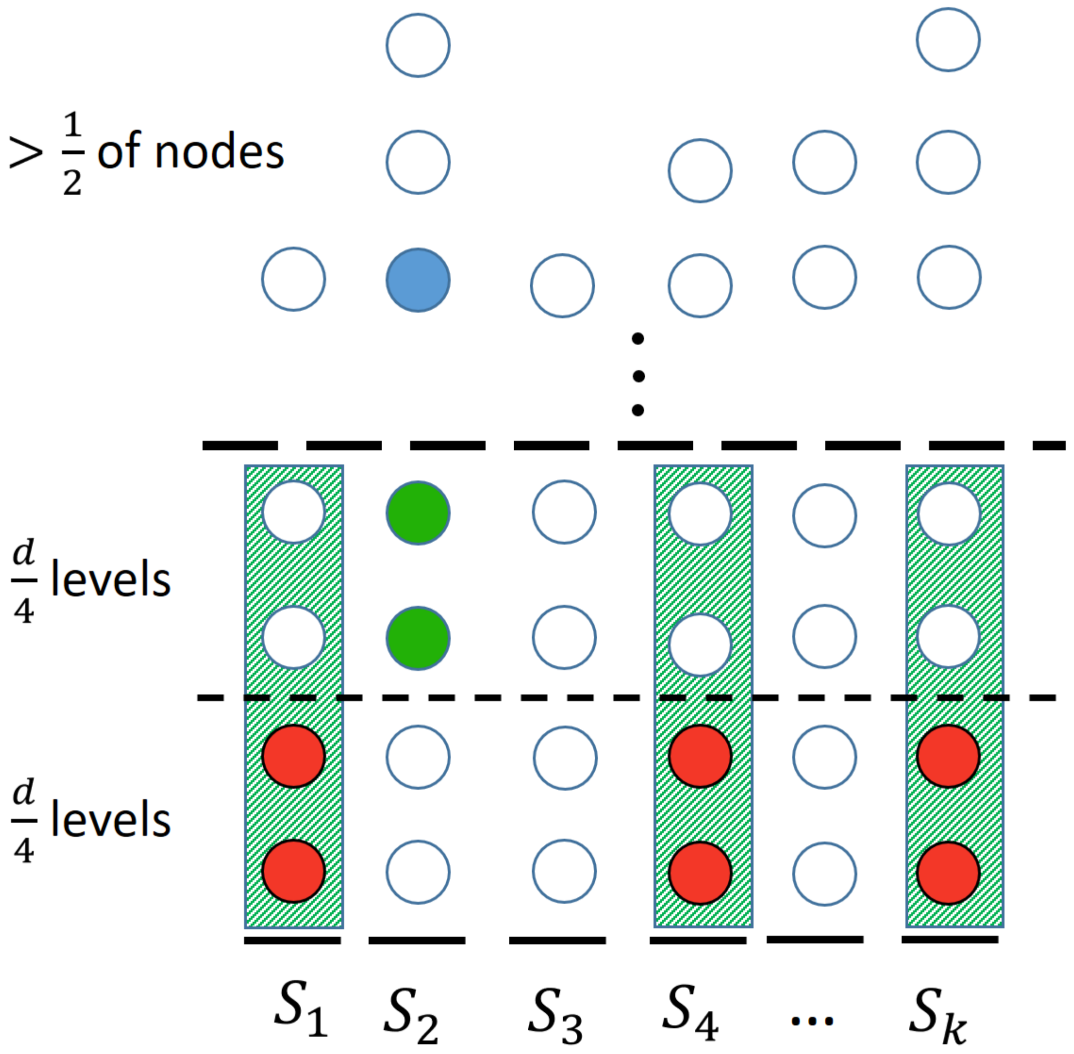
- Red node dominates $d/4$ green nodes
- Green nodes mix, so belong to $\Omega(\sqrt{d})$ paths.
- Every green node dominates at least $d/4$ in every path that it appears.
- The red node dominates $\Omega(d^{3/2})$ blue nodes.

Finding a half-Critical Node



K-mixing: Every set U intersects $\sqrt{|U|}$ of paths $P_1, \dots, P_k$

Define subchains $S_1, S_2, \dots, S_k$ such that:

- Each $S_i \subseteq P_i$.
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Claim: Most nodes dominates $\Omega(d^{3/2})$ nodes.

- Blue node dominates $d/4$ green nodes
- Green nodes mix, so belong to $\Omega(\sqrt{d})$ paths.
- Every green node dominates at least $d/8$ in every path that it appears.
- The blue node dominates $\Omega(d^{3/2})$ red nodes.

Existence of a Critical Node

Main Lem: Every k-mixing POSET with paths has a $\Omega(d^{3/2})$-critical element, where d is ”average” length of a path.

Most nodes dominate $\Omega(d^{3/2})$ nodes.

Most nodes are dominated by $\Omega(d^{3/2})$ nodes.

Therefore, there is an $\Omega(d^{3/2})$-critical node.

$\blacksquare$

Proof Summary

k - mixing POSETs have $\Omega(d^{3/2})$ critical nodes

k-mixing POSETs have at most C^k downsets

Rotation POSETs are 2n-mixing

#downsets in rotation POSETs is at most C^n

#stable matchings is at most C^n

Future directions

- Getting close to the 2.28^n lower bound?
  - Our current bound is about 2^{17n}

- Counting algorithms for estimating
  - #Stable Matchings [Dyer-Goldberg-Greenhill-Jerrum'04,Chebolu-Goldberg-Martin'12]
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Any questions?